Identical Sets of Eigenvalues

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Eigenvalues and the characteristic equations are important concepts and have many useful applications in econometrics. Hence, knowing that two matrix have identical sets of eigenvalues saves a great deal of time in finding one when the other is known. In this paper, we develop a three-part problem concerning this aspect. The first part shows that, under certain condition, matrices A and E = - A have identical sets of eigenvalues. The second part works out the solutions for nonzero characteristic roots of A. The third part discusses determinant of A. The last part discusses the application and conclusion.

INTRODUCTION

Finding a set of eigenvalues is crucial in econometric analysis. As shown in Greene (2001, p. 37), from a set of eigenvalues, we can produce complete solutions for a system of characteristic equations if the set of characteristic vectors is normalized. The set can also be used to determine rank of a matrix in panel-data technique or check the stationarity of an autoregressive process in a time-series data, or study many other properties of the series. For example, if the eigenvalues are all distinct, then the associated eigenvetors are linearly independent. This leads to many useful decompositions of a square matrix such as the decomposition into the inverse of the eigenvector-eigenvalue product or the Jordan decomposition.[1]

Another application of eigenvalues is to find out the definiteness of a matrix. For example, a symmetric matrix is positive definite if and only if all of its characteristic roots are positive, whereas a symmetric matrix is positive semi definite if and only if all of its characteristic roots are nonnegative.[2] Hence, knowing that two matrix have identical sets of eigenvalues saves a great deal of time in solving for these solutions. In this paper, we introduce a three-part problem that discussing identical sets of eigenvalues, the solutions for nonzero characteristic roots, and their properties. Section 2 of the paper states the problem. Section 3 offers the solutions, and section 4 concludes.

THE PROBLEM

It is well known that if A is a $n \times n$ matrix and G a non-singular $n \times n$ matrix, then A and $G^{-1}AG$ have the same set of eigenvalues (Magnus and Neudecker, p. 14, for example). The identity of eigenvalues between a pair of square matrices is also possible in a distinctly different context. Hence, we set up the problem as follows:

Define a real square matrix of order $n\ (= p+q)$ as

$$A = \begin{bmatrix} 0_{p\times p} & B \\ B' & 0_{q\times q} \end{bmatrix}$$

where B is a $p \times q$ matrix of $rank(B) = r$.

Part 1. Show that A and $E \equiv -A$ have identical sets of eigenvalues.

Part 2. Work out the solutions for nonzero characteristic roots of A in terms of eigenvalues of BB' or $B'B$.

Part 3. Show that if B is nonsingular, $|A| < 0\ (> 0)$ if p is odd (even).

THE SOLUTIONS

Part 1

The characteristic equation corresponding to matrix A may be written as:

$$|A - \lambda I_n| = \begin{vmatrix} -\lambda I_p & B \\ B' & -\lambda I_q \end{vmatrix} = 0$$

In view of

$$\begin{vmatrix} P & R \\ S & Q \end{vmatrix} = |Q|\,|P - RQ^{-1}S|$$

where P and Q are square matrices and $|Q| \neq 0$ (Magnus and Neudecker, p.25),

$$|A - \lambda I_n| = \begin{vmatrix} -\lambda I_p & B \\ B' & -\lambda I_q \end{vmatrix}$$

$$|A - \lambda I_n| = |-\lambda I_q|\,|-\lambda I_p - B(-\lambda I_q)^{-1}B'|$$

$$|A - \lambda I_n| = (-1)^q \lambda^q \left|-\lambda^{-1}\left(\lambda^2 I_p - BB'\right)\right|$$

$$|A - \lambda I_n| = (-1)^n \lambda^{q-p} \left|\lambda^2 I_p - BB'\right|$$

Similarly,

$$|E - \lambda I_n| \quad = (-1)^n \lambda^{q-p} \left|\lambda^2 I_p - (-B)(-B')\right|$$

$$|A - \lambda I_n| = (-1)^n \lambda^{q-p} \left|\lambda^2 I_p - BB'\right| = |A - \lambda I_n|$$

which implies that the sets of eigenvalues of A and E are identical.

Part 2

Let $\sum$ = BB'. Since $\sum$ is nonnegative definite and rank (BB') = rank (B) = r ≤ min, $\sum$ has r positive eigenvalues and (p - r) zero eigenvalues.

Further, there exists a square matrix C of order p such that

$$C'C = CC' = I_p$$
$$\text{and } C'\sum C = D = [d_i]$$

where D is a diagonal matrix consisting of eigenvalues of $\sum$ in descending order: $d_1 \geq d_2 \geq \ldots \geq d_p$.

Hence,

$$\left|\lambda^2 I_p - BB'\right| = \left|CC'\left(\lambda^2 I_p - \sum\ \right)CC'\right|$$

$$\left|\lambda^2 I_p - BB'\right| = \left|C\left(\lambda^2 I_p - D\right)C'\right|$$

$$\left|\lambda^2 I_p - BB'\right| = \left|C\right|\left|\lambda^2 I_p - D\right|\left|C'\right|$$

$$\left|\lambda^2 I_p - BB'\right| = \left|\lambda^2 I_p - D\right|\left|C\right|\left|C'\right|$$

$$\left|\lambda^2 I_p - BB'\right| = \left|I_p - \lambda^{-2} D\right|$$

$$\left|\lambda^2 I_p - BB'\right| = \prod_{i=1}^{p}\left(\lambda^2 - d_i\right)$$

$$\left|\lambda^2 I_p - BB'\right| = \prod_{i=1}^{r}\left(\lambda^2 - d_i\right)\prod_{i=r+1}^{p}\lambda^2$$

$$\left|\lambda^2 I_p - BB'\right| = \left[\prod_{i=1}^{r}\left(\lambda^2 - d_i\right)\right]\lambda^{2(p-r)}$$

Note that $\left|C\right|\left|C'\right| = 1$.

Combining the above equation and equating to zero, we obtain characteristic equation for *A* as

$$(-1)^n \lambda^{n-2r} \prod_{i=1}^{r}\left(\lambda^2 - d_i\right) = 0$$

From this equation readily follow eigenvalues of *A*:

$$\lambda = 0 \quad \text{with a multiplicity of } \ n - 2r\,;$$

$$\lambda^2 = d_i \quad (i = 1, 2, \ldots, r) \quad \Rightarrow \quad \lambda = \pm\sqrt{d_1},\ \pm\sqrt{d_2},\ \ldots,\ \pm\sqrt{d_r}$$

Part 3

If B is nonsingular, then it is a square matrix of order p = q = r. Hence, the eigenvalues of *A* are all nonzero:

$$\lambda = \pm\sqrt{d_1},\ \pm\sqrt{d_2},\ ...,\ \pm\sqrt{d_p}\ .$$

Therefore,

$$|A| = \prod_{i=1}^{p}\left(+\sqrt{d_i}\right)\left(-\sqrt{d_i}\right) = (-1)^p \prod_{i=1}^{p} d_i$$

Since $d_i > 0 \ \ (i = 1, 2, ..., p)$, $|A| < 0 \ (> 0)$ if p is odd (even) .

As a special case of *A*, let $B = I_p$:

$$A = \begin{bmatrix} 0_{pxp} & I_p \\ I_p & 0_{pxp} \end{bmatrix}.$$

Since $\Sigma = BB' = I_p$, $\lambda^2 = d_i = 1$ (i = 1, 2, …, p)

Hence, from this equation easily follows $|A| = (-1)^p$ as noted in Magnus and Neudecker (1999, p. 16).

CONCLUSION

Eigenvalue have many useful application sin econometrics. From a set of eigenvalues, we can produce complete solutions for a system of characteristic equations if the set of characteristic vectors is normalized. Additionally, the set of eigenvalues can be used to find rank of a matrix or check the stationarity of an autoregressive process. Hence, knowing that two matrix have identical sets of eigenvalues saves a great deal of time in solving for these solutions. The three-part problem introduced in this paper works out in details the general concepts of eigenvalues in a partition matrix. It is our hope that this paper will further the application of mathematics to econometric technique.

ENDNOTES

1. Please see Hamilton (1994), pp. 729-32.
2. Please see Judge at al. (1982), pp. 165-166

REFERENCES

Greene, W. (2001). *Econometric Analysis*, the Fifth Edition (Prentice Hall, New Jersey).

Hamilton, J (1994). *Time Series Analysis,* (Princeton University, New Jersey).

Judge, G., Hiil, R. C., Griffiths, W, Lutkepohl, H. and T. Lee (1982), *Introduction to the Theory and Practice of Econometrics,* (Wiley, New York).

Magnus, J.R. & H. Neudecker (1999). *Matrix Differential Calculus: With Applications in Statistics and Econometrics*. Revised Edition (Wiley, New York).