The Newsvendor Problem with Pricing and Secondary Revenues

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Many industries (e.g. hotel, rental car, cruise line and airline companies) consider secondary revenues a major source of profitability. In 2010, for instance, the five largest airlines in the United States received a total of $2.7 billion in revenue from baggage fees alone. Some casinos give away rooms since secondary activities are so profitable. Secondary revenues cannot occur without the purchase of a primary item. The price of the primary item is crucial. We consider optimal inventory levels and prices for primary items. We allow the secondary revenue to depend on the price of the primary item.

INTRODUCTION

In 2010, the five largest airlines in the United States (Delta, American, U.S. Airways, Continental and United) received a total of $2.7 billion in revenue from baggage fees alone, up from $344 million just three years prior (Bureau of Transportation Statistics, U.S. Department of Transportation). Also in 2010, hotels in the U.S. recorded revenues of over $127 billion (Hotel Operating Statistics Study, 2010, STR Global). These two industries, offering products or services that expire (e.g.; a flight on a plane or an evening in a hotel) incur costs of making those products or services available for sale. The number of items ultimately sold depends on how many items are made available for sale in the first place, how many customers are willing to buy them ah a customer will spend on secondary revenues after purchasing a primary item. These two issues (uncertain demand and a perishable product) form the basis of the newsvendor problem.

The classical newsvendor problem only considers the case of one homogeneous good. Of late, there has been a lot of interest in many industries on secondary revenue associated with a primary good. The most notable examples are in the airline industries. Indeed airlines such as Spirit and Ryan Air offer rock bottom prices for seats (the primary item) and get much of their overall revenue from secondary sources such as checked bag and seat assignment fees. Indeed, Spirit and Allegiant even charge for cabin baggage! For cut rate airlines, such secondary pricing is expected by passengers. For other airlines, where the basic item is not as drastically reduced in price, charging for seat selection, etc. can result in customer dissatisfaction. In the gambling industry, the return from secondary revenues can be so great that a vendor will give the primary item away for free. This for instance is the case for Harrah's where free rooms are provided to customers who are expected to spend a lot at the casino. Indeed, if all rooms are taken, Harrah's will often pay for a room at a hotel other than their own in order to obtain secondary revenues. (Metter's et al, 2008.)

In the classical news vendor problem, the goal is to find the optimal inventory level for a given price. Of late, there has been increased interest in jointly deriving both the price and infantry level. (See Raz and Porteus, 2006, Wilson and Sorochuk, 2009 and Wilson et al., 2011.) The first to approach the newsvendor problem with price as a decision variable is Whitin,1955.The first challenge in allowing prices to be a variable is modeling demand, where "…the demand function shows, in equation form, the relationship between the quantity sold of a good or service and one or more variables" (Samuelson and Marks, 2010). In modeling the news vendor problem, it has been commonplace to assume an arbitrary but mathematically tractable form for the expected demand function. The most common forms are linear ($a-bP$) and iso-elastic (aP^{-b}). For stochastic demand, the next step is usually to assume either linear or multiplicative uncertainty. Typically the uncertainty term is assumed to be independent of price. (For an exception, see Young,1979). Often, little justification is given for these assumptions. For a discussion of these assumptions and a comparison of the different approaches see Wilson et al., 2011. In any case, once a demand model has been chosen, solving the newsvendor problem with pricing begins in a variety of ways, depending on the conditions. Two examples are Karlin and Carr (1962) and Petruzzi and Dada (1999).

Following Wilson and Sorochuk 2009, we follow a different approach and assume that the maximum amount a randomly selected customer is willing to pay (the "reservation price ") is a random variable which can be derived from marketing studies. We model the newsvendor problem with secondary items and, in addition to the inventory level, we the primary price to be decision variable. The modeling of secondary revenues is starting to generate interest in the academic community. For instance see Allon et al., 2011 and Shulman and Geng, 2012. Early papers to consider this include Fort (2004), Marburger (1997) and Rosenfield (1997).

We start by briefly describing the classical news vendor problem. Then we summarize our approach to modeling demand where we make no a-priori assumptions about the mathematical form of demand functions. Our approach to modeling secondary revenues is then described and we provide expressions for the expected value and variants of total profit for given levels of inventory level and price. We then provide a number of examples. First, we consider the distribution of profit values as parameters (such as the expected value of the secondary revenue from a customer) are varied.

THE NEWSVENDOR PROBLEM

In the basic form of the newsvendor problem, a seller of an inventory of a homogeneous product attempts to maximize expected profit by deciding on an inventory quantity Q, before knowing exactly how many customers will want to buy an item at an exogenous price, P. For each unit of inventory the seller decides to make available for sale, a constant marginal cost of c is incurred ($P>c$), regardless of whether or not the item is ultimately sold. Although the newsvendor problem has been expanded to include the possibility of salvage value for unsold items, here it is assumed that the items are worthless if not sold at full price. Denote the number of customers willing to pay price P for an item as random variable $X(P)$ with corresponding cumulative distribution function $F_{X(P)}(\cdot)$. Given these conditions, the profit-maximizing quantity, Q^*, is determined by the well-known fractile solution:

$$Q^*(P) = F_{X(P)}^{-1}\left(\frac{P-c}{P}\right),$$

where $F_{X(P)}^{-1}(\cdot)$ denotes the inverse cumulative distribution function of demand. (For the case where Q is only allowed to take on discrete values, $Q^*(P)$ is the smallest integer value of Q that is greater than or equal to the right-hand side of the above expression.)

GENERAL MODEL OF DEMAND

Consider demand as a choice model where each customer in the population has a choice – buy or not buy one item that is available for sale. Here, individual consumer behaviour is modelled and aggregated into the demand for the product.

The size of the customer base interested in the product is provided by a random variable *D* (or *d* in the case of a deterministic customer base). A randomly selected customer has a reservation price represented by the random variable *RP* whose cumulative distribution function will be denoted by $F_{RP}(\cdot)$. This assumption captures a wide range of realistic behaviours. We do not require a 'low before high" assumption and assume that customers arrive at random. These assumptions lead directly to appropriate demand equations without the need to assume specific mathematical forms for expected demand. (A discussion on the use of this approach versus assuming specific mathematical demand equations can be found in Wilson and Sorochuk, 2009)

Given this framework, demand, the number of customers willing to buy an item at price *P*, is a random variable, and is denoted *X*(*P*). For a deterministic customer base, *d*, the random demand follows a binomial distribution with probability mass function, $h_{X(P)}(\cdot)$, given by

$$h_{X(P)}(x) = Prob\big(X(P) = x\big) = \binom{d}{x}\big(1 - F_{RP}(P)\big)^{x}\big(F_{RP}(P)\big)^{d-x},\ x = 0, 1, \dots d\ .$$

(The modification for the case where the number of customers is random is to condition the above on the event *D*=*d*, multiply by *P*[*D*=*d*] and sum over *d*.) The cumulative distribution function, $F_{X(P)}(\cdot)$, is given by $F_{X(P)}(x) \equiv \sum_{y=0}^{x} \binom{d}{y}\big(1 - F_{RP}(P)\big)^{y}\big(F_{RP}(P)\big)^{d-y}$.

Note that the normal approximation to the binomial distribution can be used to model *X*(*P*) as a normal random variable with mean, and variance given by $\mu_{X(P)} = d\big(1 - F_{RP}(P)\big)$ and, $\sigma^2_{X(P)} = d\big(1 - F_{RP}(P)\big)\big(F_{RP}(P)\big)$, respectively--provided both parameters are greater than five. Thus, even for large values of *d*, computations are not unduly complex

SECONDARY REVENUE

Secondary profits can be realized only after a customer has purchased a primary item. We represent the secondary revenue obtained from a given customer by the random variable *S*(*P*), where *P* is the price of the primary item. This formulation allows for the study of a number of realistic situations. There will be cases where *S*(*P*) does not actually depend on *P* and will be denoted simply by *S*. For instance, for a range of prices at a ball game, one could reasonably expect that the amount a customer spends on beer is unrelated to the admission price. For casino hotels, it can be reasonable to suppose that the amount many customers will gamble is not very related to the room rate. There are also situations where a customer has a total budget and anything spent on the primary item will reduce the amount spent on secondary items. One can think of a family of four, say, with a fixed budget going to Disneyworld—too expensive an admission price will reduce the amount available for secondary purposes and might, indeed, preclude a family going at all. Then there are cases common in the marketing of high priced items such as cars. There a high primary price can make a customer less reluctant to buy extras such as racing stripes since, even though objectively the racing stipe might cost a lot, its price pales in comparison to the tens of thousands spent on the car.

THE PROFIT FUNCTION

Let $Z_1(P,Q)$ denote the number of primary items that are sold if the inventory of primary items is Q and the price of a primary item is P, i.e. $Z_1(P,Q)=\min(X(P),Q)$. Let $\Pi_{Total}(P, Q)$ be the random variable that gives the total profit. Then, the expected total profit for a seller who receives revenue from the sale of secondary items in addition to the revenues from the sale of primary items is

$$E[\Pi_{Total}(P, Q)] = E[Z_1(P,Q)]\big(P + E[S(P)]\big) - cQ .$$

Comparing the above expected profit of formulation to that of the classical news vendor problem, the optimal order quantity for a given price P is given by

$$Q^*(P) = F^{-1}_{X(P)}\left(\frac{P + E[S(P)] - c}{P + E[S(P)]}\right).$$

Expected profit is not always the most important criterion for Newsvendor problems. If it is a one-shot event (a special concert for instance) rather than a repeated serious of problems (e.g., the same flight leaving every day), then the distribution of possible profit values—in particular the standard deviation or variance—becomes important. The variance in total profit for a seller who receives profit from the sale of secondary items in addition to the revenues from the sale of primary items is given by

$$Var\big(\Pi_{Total}(P, Q)\big) = Var\big(S(P)\big)E[Z_1] + \big(E[S(P) + P]\big)^2 Var\big(Z_1(P,Q)\big).$$

(Derivations for the expected value and variance can be found in the Appendix.)

EXAMPLES – SECONDARY REVENUE INDEPENDENT OF PRICE

To demonstrate the sensitivity of the profit distribution to certain variables in the model, some numerical examples are provided. In each example, one of the parameters needed to calculate profit is varied, while the others are held constant. For a given set of values, demand (the number of customers willing to pay price P for a primary item) as well as total profit from secondary items are randomly generated and used to calculate profit. The process is repeated 100,000 times for each set of values, and the results to show the relative frequency of total profits. This provides an empirical probability function for the random variable $\Pi_{Total}(P, Q)$.

For these examples, the size of the customer base is fixed at $d = 50$ and the cost of making each primary item available for sale at $c = 10$. The maximum amount, RP, that a randomly-selected customer is willing to pay for an item is random and follows a normal distribution with a mean of $E[RP]$ and a standard deviation of 10, i.e. $RP \sim N(E[RP], \sigma_{RP} = 10)$.. The secondary item revenue received from a customer who has already purchased a primary item is assumed independent of P and follows a normal distribution given by $S \sim N(E[S], \sigma_S = 10)$. The number of primary items made available for sale by the seller is $Q = 40$.

Sensitivity to Changes in Expected Reservation Prices

Here we set P and $E[S]$ equal 50. Figure 1 shows the sensitivity of the total profit distribution to the mean reservation price, $E[RP]$, as it varies from 30 to 60 in increments of 10.

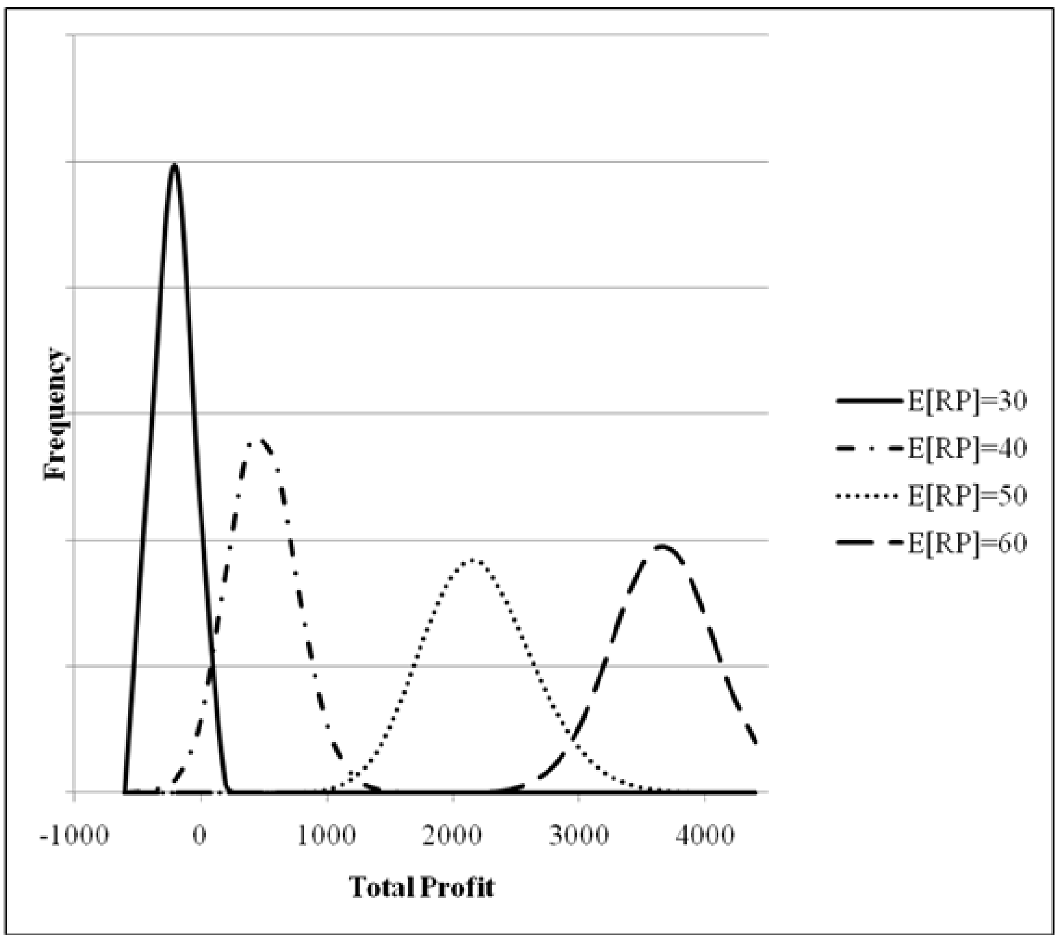
Naturally, as the expected reservation price increases, so does the likelihood of selling more primary items (and gaining secondary profits as well). As expected demand for items increases, either more of the available items are sold or they are all sold and there is unmet demand. The change in expected total profit is not the same from plot to plot even though the change in expected reservation price is constant between plots. For an expected reservation price of 30, the expected profit is negative. However, a small fraction of the time it is positive. Consider how variance in total profit changes as reservation price increases. Recall the expression for variance:

$$Var\left(\Pi_{Total}(P,Q)\right)=Var(S)E[Z_1]+\left(E[S+P]\right)^2 Var(Z_1).$$

Although it appears at first glance that variance should increase continually as more customers are willing to buy an item, if the expected reservation price is high demand will often exceed supply and stock-outs will occur which means a smaller variance in the number of items sold.

This phenomenon is seen in the plot for an expected reservation price of 60 where the spread of values is much smaller than for the other values for expected reservation price.

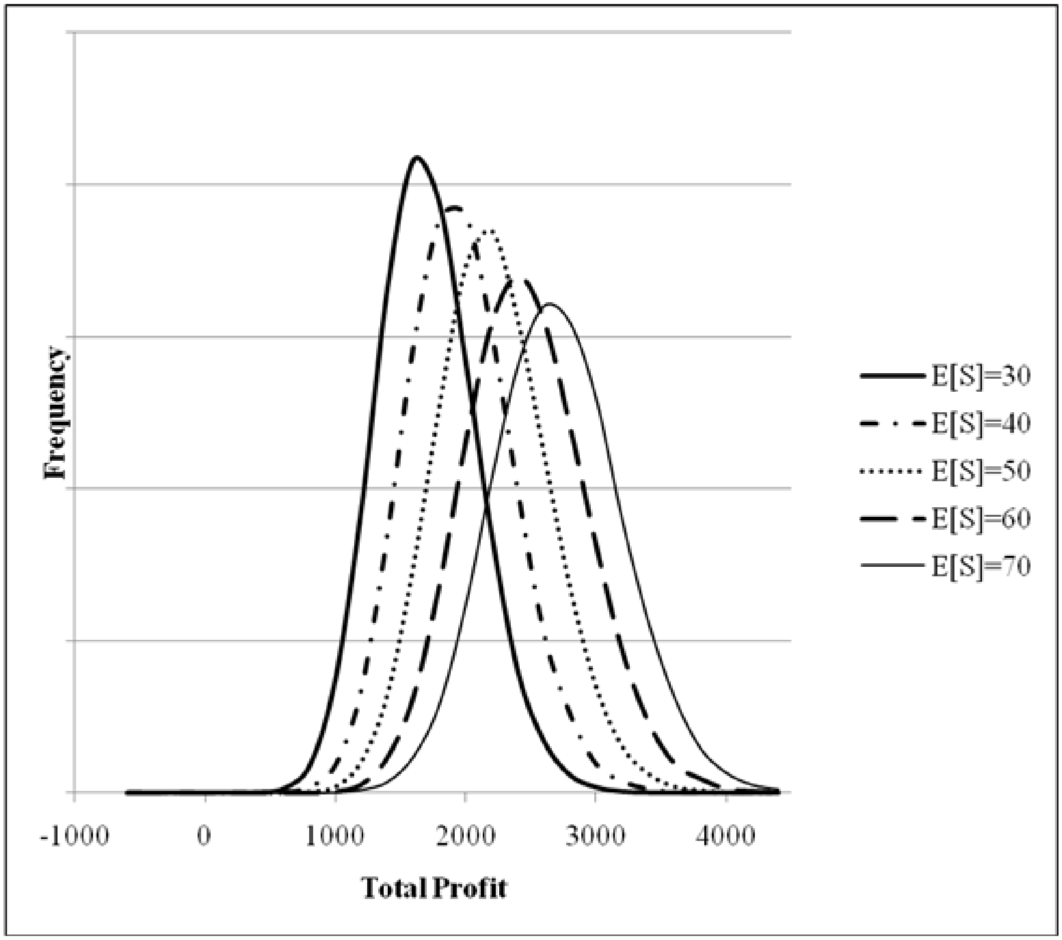
FIGURE 1
SENSITIVITY OF TOTAL PROFIT TO CHANGES IN EXPECTED RESERVATION PRICE



Sensitivity to Changes in Expected Secondary Revenues

In Figure 2, P and $E[RP]$ are set equal to50. The expected value, $E[S]$, a randomly selected customer will pay for secondary items is varied from 30 to 70 in increments of 10.

FIGURE 2
SENSITIVITY OF TOTAL PROFIT TO CHANGES IN EXPECTED SECONDARY PROFIT



The effect of different secondary profits is not nearly as large as those seen in Figure 1. While increasing the expected secondary profit does increase the variance and shift the profit function in the positive direction, the overall effect of increased expected profit is small.

This can be explained using the expression for expected profit:

$$E[\Pi_{Total}(P, Q)] = E[Z_1](P + E[S]) - cQ\,.$$

The expected total profit increases linearly with expected secondary profit, up to the point where no more customers can purchase a primary item (and therefore contribute secondary profit to the seller).

The increased variance in total profit seen in Figure 2 is explained by the $E[S+P]$ term in the expression for variance

$$Var(\Pi_{Total}(P, Q)) = Var(S)E[Z_1] + (E[S+P])^2 Var(Z_1).$$

As expected secondary profit increases, so does the right hand side of the above expression.

Sensitivity to Changes in the Selling Price

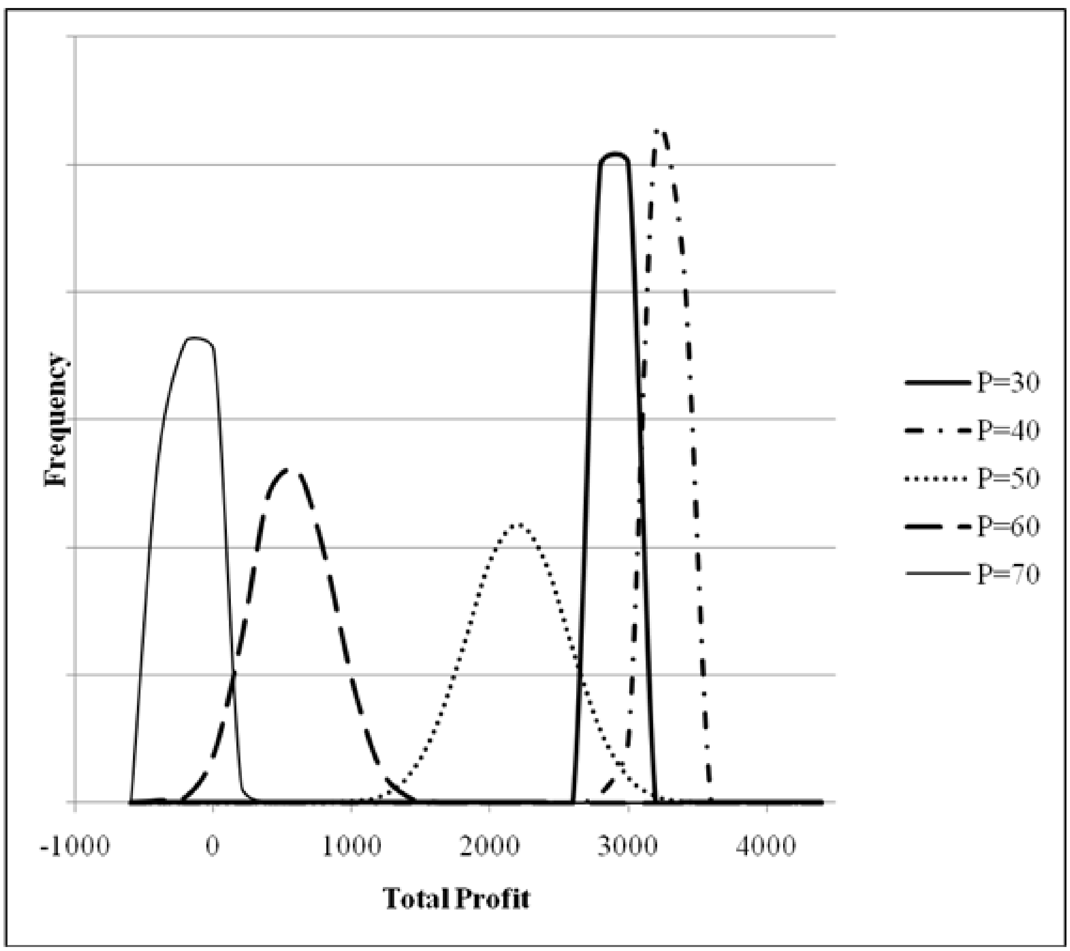
In Figure 3, five functions demonstrate the effect of different selling prices on profit. Here, the selling price P takes on values of 30, 40, 50, 60 and 70 while $E[S]$ and $E[RP]$ equal 50. At a low selling price (P = 30), profits are high since many customers are willing to pay even P = 40 for a ticket. The result is not only high revenues from ticket sales, but as a result more customers available to purchase secondary items. At a high selling price (P = 60), profits are low, as fewer customers are willing to pay P = 60 for a ticket. The result is not only low revenues from ticket sales, but fewer customers available to purchase secondary items. At some point increasing the price will reduce the demand and an increase in P will not be enough to offset the revenue lost from fewer customers and less secondary revenue.

Increasing the selling price increases the $E[S+P]$ term on the right-hand side of the expression for variance:

$$Var(\Pi_{Total}(P,Q)) = Var(S)E[Z_1] + (E[S+P])^2 Var(Z_1).$$

However, if primary items are priced such that very few are willing to purchase everyone is then the variance in profit is mostly due to the variance in secondary profits since $Var(Z_1)$ is then small.

FIGURE 3
SENSITIVITY OF TOTAL PROFIT TO CHANGES IN SELLING PRICE



EXAMPLES – SECONDARY REVENUE DEPENDENT ON PRICE

In this section, we allow the secondary revenues to depend on Price specifically, we assume that the expected value of secondary revenue is linearly related to price, i.e. $E[S(P)] = a+bP$, for some a and b. This can encompass a number of realistic situations. For instance, the situation where a customer has a total budget of a is covered by taking b equal to -1. The case $b>0$ allows for the effect (commonly seen for customers buying high-priced items such as cars) where a customer is willing to pay more for secondary item as the primary price increases. The case $b<0$ models the situation where a customer will pay less for a secondary item as the primary price increases.

In Figure 4, we consider the case where the possible number of customers is 25, a=5 and the reservation price is normally distributed with a mean of 10 and a standard deviation of 3. Here we allow the cost of an item to take on values of 1, 5, 10 and 15 and look at how optimal inventory levels, optimal prices and optimal profits vary as a function of b. These optimal values are obtained by maximizing the expected profit function given previously.

FIGURE 4(A)
OPTIMAL PRICES FOR $RP \sim N(10, 3)$, $d = 25$, AND $a = 5$

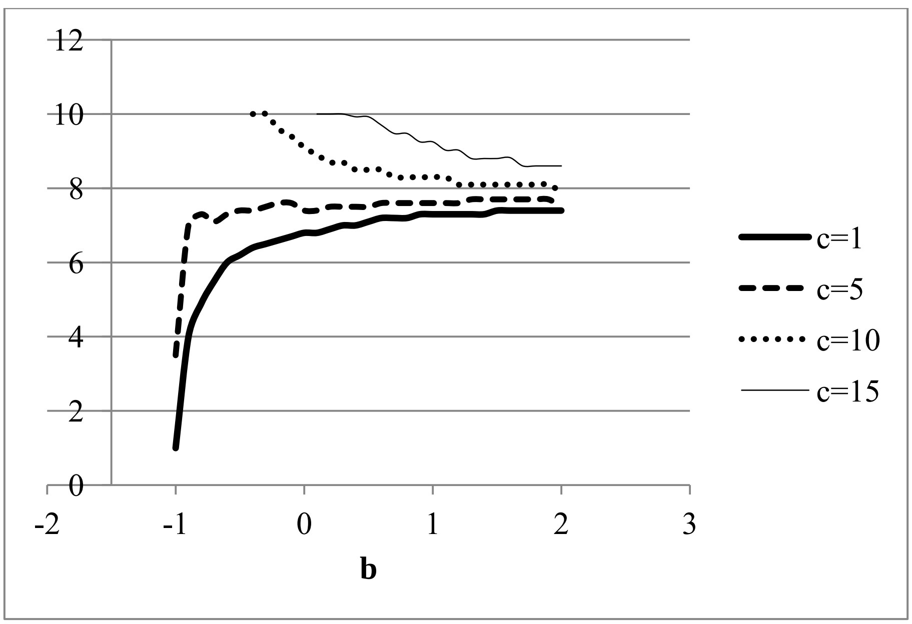


FIGURE 4(B)
OPTIMAL INVENTORY LEVELS FOR $RP \sim N(10, 3)$, $d = 25$, AND $a = 5$

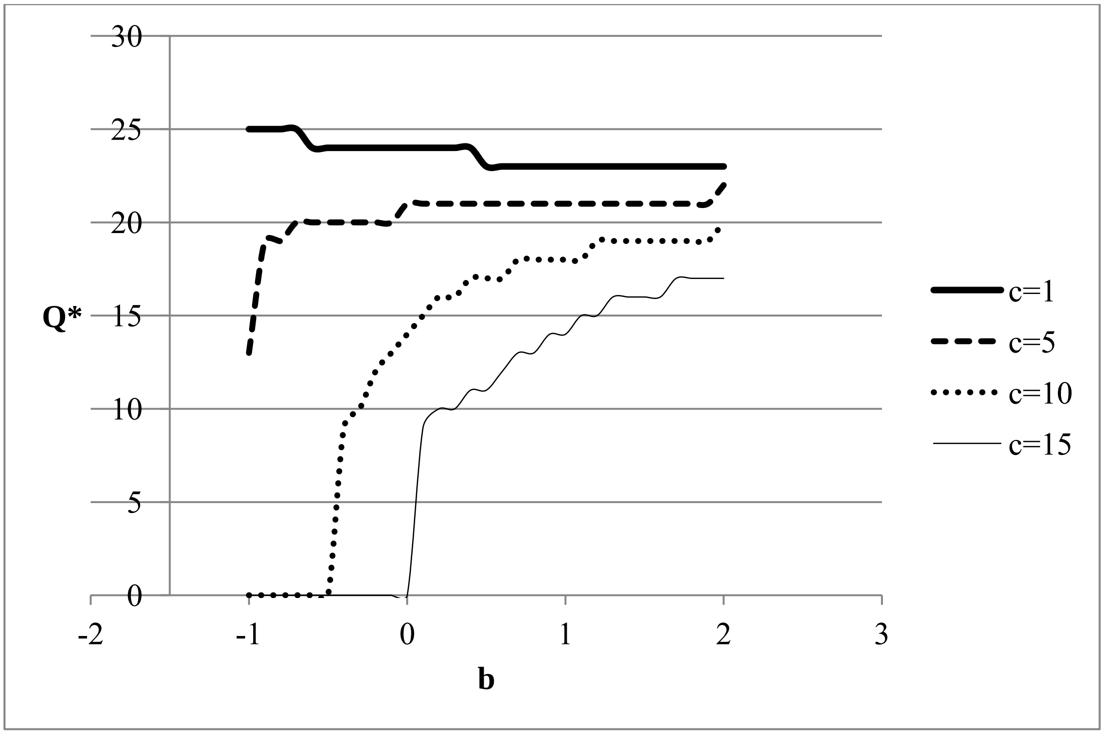
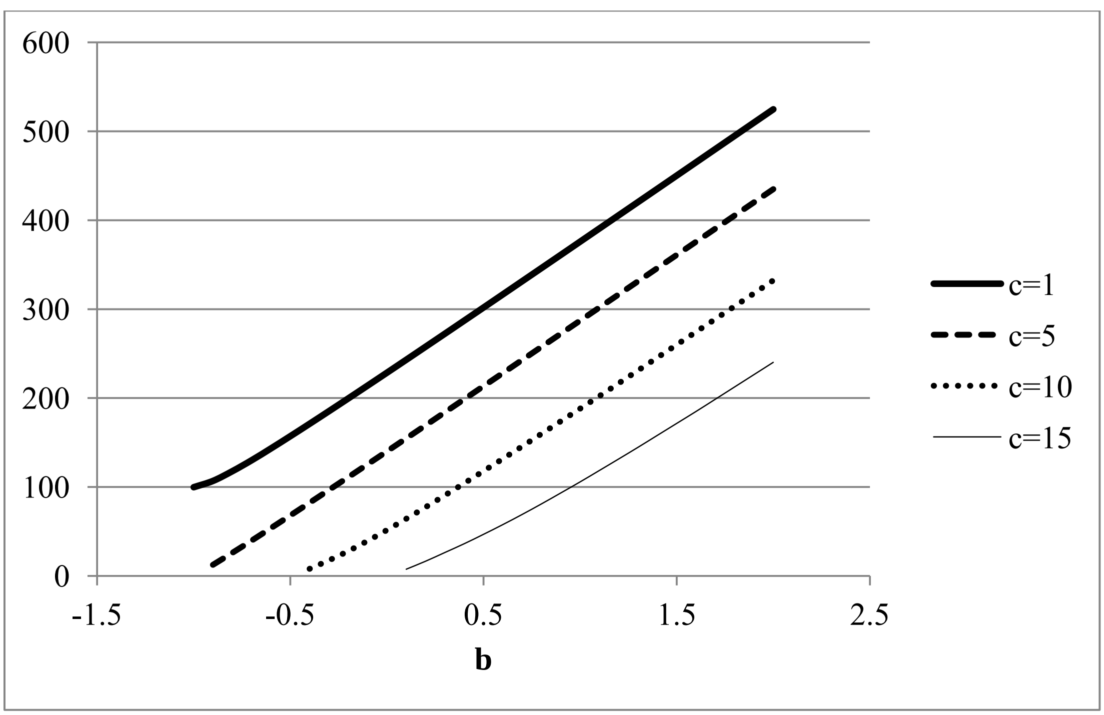


FIGURE 4(C)
OPTIMAL EXPECTED PROFITS FOR $RP \sim N(10, 3)$, $d = 25$, AND $a = 5$



As expected and seen in Figure 4(c), optimal expected profits increase as *b* increases and *c* decreases. (The cases where expected profit is negative are excluded from the graphs.) Let's consider the case where you the cost of the primary item is small (*c*=1, c=5). From Figure 1(c), at *b*=-1, the optimal price is very low. As discussed above, this case corresponds to the situation where the expected total amount of customers willing to pay for both primary and secondary items is fixed (here at *a*=5). So here, ideally, prices should be set low in order to maximize the number of people who will buy the primary item. (Recall that secondary revenue only occurs when the primary item is purchased.) Consider the case where *c*=15. Here the optimal price at *b*=-1 is so low relative to the reservation price has to be practically equal to zero. (Almost all customers will buy at this price since the expected reservation price is 10 with a standard deviation of 3.) For high fixed costs (*c*=10 or *c*= 15), it is not optimal to offer items since expected total budget per customer of five is not sufficient to cover costs. This situation can also be seen from Figure 4(b). At *b*=-1 and *c*=1, it is optimal to stop as many items as there are possible customers. At *b*=-1 and *c*=5, fewer items will be stalked since the item cost *c* is higher. For the two higher values of *c*, the optimal inventory level is zero. For high item costs, as *b* increases, the optimal prices becomes smaller and smaller in an effort to obtain more customers willing to pay the primary price since the secondary revenues go up for these customers as a function of primary item price.

In Figure 5, the reservation price now has a mean of five, a standard deviation of 1.5 and *a* = 10. Parts (a) and (b) give the optimal prices and infantry levels, respectively, while part (c) gives the optimal profits. The plots exhibited similar behavior to those in Figure 4.

FIGURE 5(A)
OPTIMAL PRICES FOR $RP \sim N(5, 1.5)$, $d = 25$, AND $a = 10$

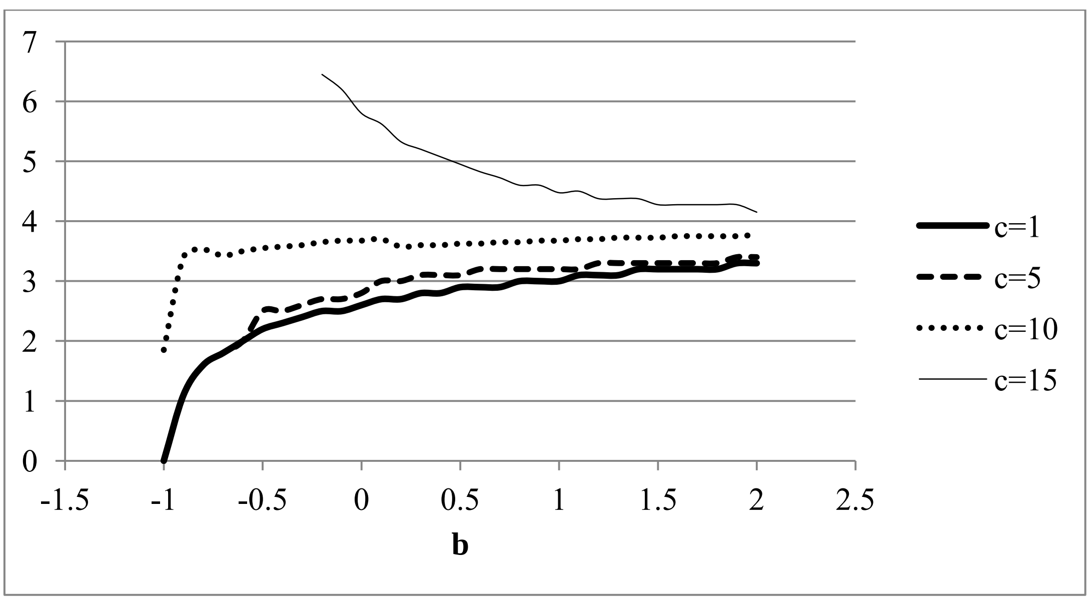


FIGURE 5(B)
OPTIMAL INVENTORY LEVELS FOR $RP \sim N(5, 1.5)$, $d = 25$, AND $a = 10$

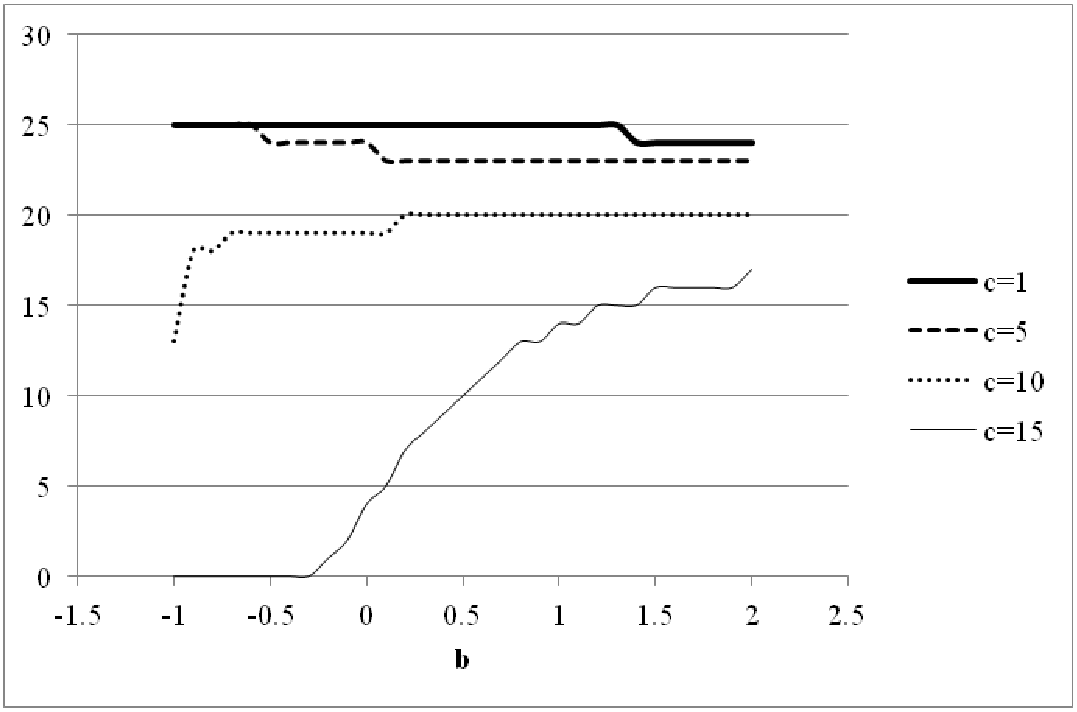
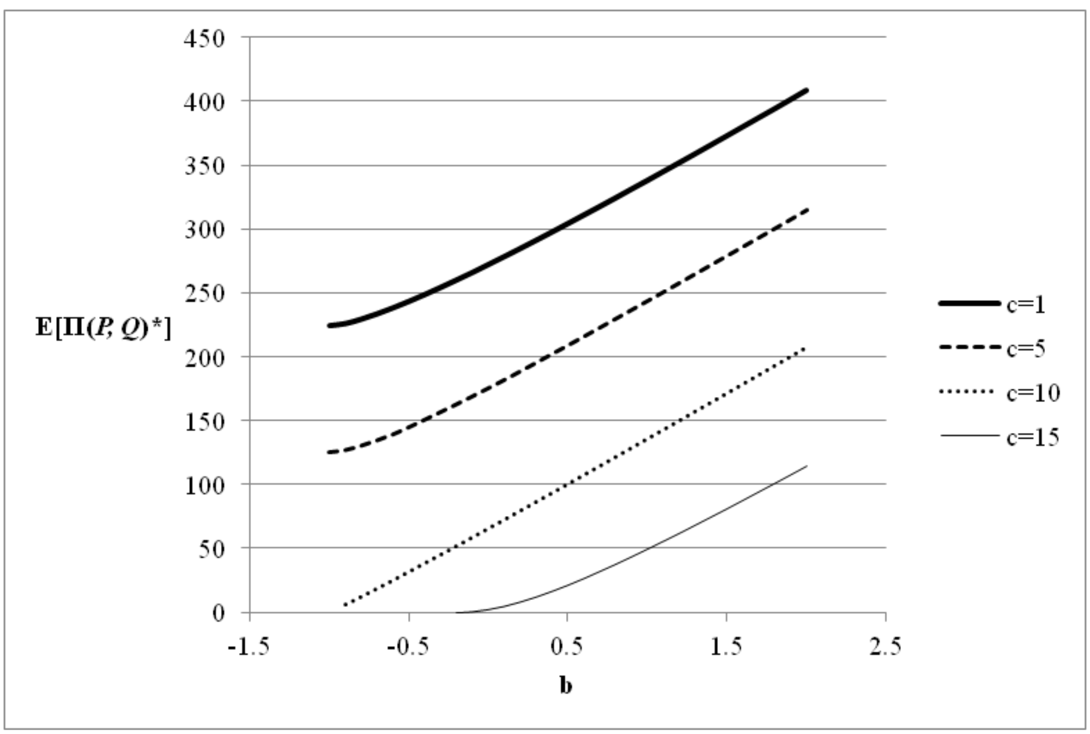


FIGURE 5(C)
OPTIMAL EXPECTED PROFIT FOR $RP \sim N(5, 1.5)$, $d = 25$, AND $a = 10$



CONCLUSION

In this paper, we have shown modeling the effect of secondary revenues can be accomplished by modifying the basic newsvendor problems. In some circumstances, it can make sense to heavily discount the primary item in order to achieve maximum secondary revenues. Such modeling becomes increasingly more important as industries adopt models where much of their profit is obtained through secondary revenues. "Carnival, for instance, recently reported that 'onboard other revenue' jumped from$936 million in the third quarter of 2011 to $965 million in the third quarter of this year. Meanwhile, ticket revenue declined from $3.9 billion to $3.6 billion over the same period. (See Trejos, 2012.)

APPENDIX: EXPECTED VALUE AND VARIANCE OF PROFIT

First, we demonstrate that for seller of homogeneous items who also receives a random amount of secondary revenues for each item sold, the expected profit is given by $E[\Pi_{Total}(P,Q)] = E[Z_1(P,Q)]\left(P + E[S(P)]\right) - cQ$, where $Z_1(P,Q) = \min\left(Q, X(P)\right)$ denotes the number of primary items that are sold and $E[S(P)]$ denotes the expected revenue from secondary items.

The profit can be written as

$$\Pi_{Total}(P,Q) = P\min\left(Q, X(P)\right) + \sum_{i=1}^{\min\left((Q,\, X(P)\right)} S_i(P) - cQ = PZ_1(P,Q) + Z_2(P,Q) - cQ,$$

where $Z_2(P,Q)$ denotes the amount $\sum_{i=1}^{\min\left((Q,\, X(P)\right)} S_i(P)$ received from secondary sales. The expected revenue from secondary items, $Z_2(P,Q)$, is the random sum of random variables and, from Wald's lemma, has an expectation equal to $E[Z_1(P,Q)]E[S(P)]$. Thus, taking expectations the expected profit can be written as

$$E[\Pi_{Total}(P,Q)] = PE[Z_1(P,Q)] + E[Z_1(P,Q)]E[S(P)] - cQ.$$

Now justify the formula for the variance of total profit. Note that the profit can be written as follows:

$$\Pi_{Total}(P,Q) = \sum_{i=1}^{Z_1(P,Q)} \left(P + S_i(P)\right) - cQ.$$

From the Law of Total Variance,

$$\begin{aligned} Var\left(\sum_{i=1}^{Z_1(P,Q)} \left(P + S_i(P)\right) - cQ\right) &= Var\left(S(P) + P\right)E[Z_1(P,Q)] + \left(E[S(P) + P]\right)^2 Var\left(Z_1(P,Q)\right) \\ &= Var\left(S\right)E[Z_1] + \left(E[S + P]\right)^2 Var\left(Z_1\right), \end{aligned}$$

as required.

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