Fuzzy Liability Driven Pension Fund Management

Ilona Shiller
University of New Brunswick

Ishmael Radikoko
University of Botswana

Due to the several factors, the applicability of crisp techniques to the pension debt valuation and fund management is questionable. We take a fuzzy approach and use triangular fuzzy numbers (T.F.N.s) to capture the above imprecision. We suggest that the best pension fund strategy is that of a fuzzy contingent immunization. This will involve an active management of pension plan's assets until the value for the pension threshold is reached, at which time the pension fund manager should switch to a fuzzy immunization strategy. We apply this strategy to the Canadian firm-level data and report results.

INTRODUCTION

This study investigates liability-driven investment strategy for pension plan funding under fuzzy environment. We show that pension debt is dissimilar in nature to traditional debt. Pension debt is not easily quantifiable/tradable, cheaply retirable, or transferrable. Frequently, claimants to future plan benefits are unable to embed restrictive pension debt covenants into the pension fund governance, meaning that there are no legal restrictions put in place against pension risk-taking by pension managers. Due to the ambiguities in pension debt valuation, it's inherently imprecise nature, and an inability for an exact assessment of inputs (i.e., pension benefit requirements, appropriate discount rates, due dates for pension benefit payments, etc), the applicability of crisp techniques to the pension debt valuation and fund management is questionable. Under crisp pension asset-liability valuation/management, decisions are taken on the basis of exact parameter utilization in deterministic formulas. We take a fuzzy approach and use triangular fuzzy numbers (T.F.N.s) to capture the inherent variable / state imprecision and to effectively tackle the inefficiency of the traditional crisp pension fund management.

The US Pension Benefit Guarantee Corporation (PBGC) took over pension benefit obligations of many firms in the past and is currently running a deficit. Rauh (2006) reports that PBGC deficit reached $22.7 billion by the 2005 year-end. PBGC views this deficit as a result of poor stock market performance and low levels of voluntary contributions, in excess of PBGC legal contribution requirements. Clark and Monk (2006) note that over the past three decades 3,600 pension liabilities were taken over by the PBGC, whereas the rate of retiree defined benefit coverage fell from 80 to 45% over the 1980 to 1998 with some industries being affected more than others (i.e., steel, automotive, airline). The situation is Canada is better but the retiree funding is not sustainable at current contribution rates and is mostly due to losses incurred on equity positions. Canada Pension Plan Investment Board (CPPIB) manages Canada's pension plan (CPP) assets. CPPIB began to pursue aggressive investment strategies by investing in equity of private and public companies since 1999. Cooke (2003) reports that, over the 2002, CPP lost $3 billion on the stock market, which is equivalent to -15.9% returns on invested capital. The bear market of the early 2000s and falling interest rates had a negative impact on many pension funds' funding statuses, a positive impact on funds' contribution rates, and a negative impact on sponsor firms' overall performance. All above factors called for the government intervention to re-vitalize publicly- and privately based pension plans and to secure the benefit payments owed to the plan members.

Pension protection act (PPA, 2006) imposed new funding rules on pension funds, while the financial accounting standards board (FASB) and the government accounting standards board (GASB) imposed new accounting rules the same year. After the release of the PPA, some sponsors decided to restrict benefits, terminate their plans, close their plans to new hires, or freeze accruals for working employees. Others changed the nature of their pension asset investments to deal with greater volatility: privately-based pension plans started adopting the more conservative strategies, whereas publicly-based pension plans – the more aggressive strategies (MacDonald, 2007).

Clark and Monk (2006) view the current US pension crisis to be the result of actuarial shortcomings, risky pension asset investment strategies taken in isolation from accounting for the funded status of the pension plan and of a lack of proper regulation/accounting for pension assets and liabilities. After PPA and the introduction of new accounting rules, many plans shifted their focus from the asset-driven pension fund management to the liability-driven pension fund management. With the former, the pension fund manager sets the target for the return on pension assets and undertakes the portfolio asset allocation with the purpose of exceeding that target. With the latter, the pension fund manager minimizes pension funding costs by benchmarking the portfolio return versus the liabilities. The recent popularity of the liability-driven pension fund management stems from its ability to make asset allocation decisions simultaneously with a proper attention given to the liabilities of the pension fund to its inactive employees. It focuses on the primary objective of the pension funding – to meet projected benefit payments owed. At the same time, it does not restrict the management of the fund to passive strategies only. The fund manager may still actively manage surplus pension funds to exceed the return required to meet projected benefit payments.

This study looks at the fuzzy liability-driven pension fund strategy that combines passive management of fund assets using fuzzy immunization and active management of surplus pension assets using triangular fuzzy representation of fair and market bond prices. Using fuzzy immunization, the pension fund manager is able to account for mismatches in the interest sensitivities of assets and liabilities. We allocate pension assets only in bonds to match with the bond-like nature of firm pension benefit liabilities. We also compare the fuzzy fair bond value calculated using credit-spread adjusted Government of Canada term structure of spot rates to fuzzy quoted market prices.

The remainder of this study is organized as follows. Section 2 discusses past and current practices in pension plan management. The motivation for the utilization of the fuzzy framework in an application to the pension plan management is summarized in Section 3. The latter section also describes the data set used. Section 4 suggests an optimal contingent-immunization pension plan management strategy and briefly formulates it using fuzzy logic. It also applies fuzzy liability-driven investment approach to the data and reports results. The final section contains the conclusion.

PAST AND CURRENT PRACTICES IN PENSION PLAN MANAGEMENT AND THE ATTRACTIVENESS OF THE ASSET-LIABILITY PENSION PLAN MANAGEMENT

Pension funds channel the contributions of plan members to the beneficiaries of retirement benefits. Pension funds are frequently large institutional investors and pension fund asset allocation decisions have important implications for the funding statuses, future contribution rates, possibilities to enjoy contribution holidays, retiree benefits' recovery rates, and abilities to meet projected pension liability payments. In US, pension assets accounted for $15 trillion in 2003 (Anonymous, 2006). In Canada, they accounted for $54.8 billion in 2002 (Cooke, 2003).

This study considers the management of defined benefit obligations (DBO), defined as the present value of benefits calculated using employee current service and projected future wages at retirement (Kopcke (2006) note that accrued benefit obligations (ABO) are used on firms' accounting statements instead of projected benefit obligations (PBO). ABO represent the present value of all benefits expected to be collected by firm retirees as a function of their current wage levels and service, whereas PBO represent the present value of all benefits that also take into account future wage increases. In this study, we look at PBO, consistent with the Kopcke's statement that funding PBO is preferred to funding ABO. This is because the firm is able to minimize the variability of its aggregate pension expenses through the stabilization of contributions per dollar of wages over the years of pension fund existence. By funding PBO, the firm encourages pension funds to hold surplus pension assets and to minimize the risks of pension plan underfunding.). Defined benefit pension plan is a trust fund for employees with benefits determined by the benefit formula, which depends on employee's age, and service. Defined benefit obligations (DBO) guarantee minimum benefits payable to employees in retirement with a contingent claim on the sponsor that reflects cost of living adjustments and adjustments for tenure. For example, retirees may enjoy ad hoc benefit increases at the discretion of the management to reflect abnormal returns on pension assets or better financial performance of the sponsor firm. Besides a short call option on the value of firm's assets, sponsors also hold a long put option on the value of firm's assets with the strike price equal to the dollar pension accrued benefits guaranteed by PBGC. The embedded put option served as the primary explanation for pension fund risk-taking, meaning that firms with underfunded pension plans tended to make riskier asset allocation decisions than firms with overfunded pension plans. The latter explanation is consistent with the findings of Rauh (2006) that firms at higher risk of default tend to increase the volatility of plan's assets. In the absence of PBGC insurance and given an appropriate regulatory/accounting base, firms may want to fully fund their accrued pension obligations and invest in safer securities such as government and corporate bonds. In line with this hypothesis, Rauh (2006) finds that financially- and liquidity-constrained firms tend to allocate pension fund assets to safer securities, implying that the risk management incentives for pension plan risk-taking dominate the risk-shifting incentives. This is also consistent with the game-theoretic explanation of Ambachtsheer (2002), who shows that the greater the deviation of the pension fund funding status from the equilibrium fully-funded target level, the greater the incentive for the pension fund stakeholders to "win the surplus or win the right not to own the deficit." The agency problem between shareholders and plan members is mitigated when the pension fund funding levels come closer to the equilibrium fully-funded target level.

Inkmann and Blake (2007) discuss past and current pension regulatory and accounting frameworks. They distinguish between changes in accounting for pension assets and liabilities before and after the issuance of the US FASB's statement N 87 (FASB87), and after the introduction of the PPA in 2006 together with FASB new accounting standards. In terms of liability valuation, before FASB87, the choice about the magnitude of the reported market value of plan liabilities was based on the tradeoff between the tax benefits resulting from reporting greater market values of pension liabilities and the benefits in the form of higher accounting profits reported in annual reports arising from reporting lower market values of pension liabilities. After FASB87, plan sponsors could still make choices with respect to the magnitude of reported pension liabilities strategically, through their choice of a discount rate applied to liabilities (i.e., considering the tradeoff between benefits from a lower/higher discount rate). The discount rate may be chosen strategically due to the existing conflict of interest between sponsor firm's shareholders and plan members. The pension fund manager has an objective to maximize the shareholder's wealth and, at the same time, a duty to maximize the value of pension assets (i.e., through exceeding or at least meeting the costs of funding the pension plan). In practice, it is difficult to maximize the shareholder's wealth and the value of pension assets at the same time. This is because increasing contributions today means taking away funds available for financing profitable investment projects and vice versa. This temporal tradeoff between overfunding today and underfunding tomorrow may explain the existing tendency of mature companies with good earning prospects to overfund pension plans, and of young, start-up, companies with poor earning prospects to underfund their pension funds. Practically, after FASB87, firms would frequently use the Moody's AA-rated long-term corporate bond yield curve to discount projected pension benefits. With the PPA (2006), firms could no longer use strategic management considerations with respect to their discount rate choice because the act obligated sponsors to use monthly IRS specified modified discount rate comprising three segment rates, averaged across the three top quality rating groups and across three relevant maturity spectrums: short-term, medium-term, and long-term. Moreover, the act required sponsors to determine the funding target in accordance with the IRS published mortality tables and fund 100% of the present value of all benefits accrued today, based on current employees' salary. Restrictions were put in place with respect to the time period over which pension assets may be smoothed (24 months versus 5-year smoothing before), the time period over which accrued pension liabilities may be amortized (7 years versus up to 30 years before), increases in benefits and/or accruals. New accounting standards introduced by FASB and GASB in 2006 emphasize transparency in accounting for pension assets and liabilities. Instead of reporting pension assets/liabilities in the footnotes to the financial statements, firms are now required to report this information on their balance sheets, using marked to market accounting. This leaves less possibility for market participants to misjudge the pension plan's funding status and less possibility for managers to manipulate firm's earnings to the shareholders' advantage (Franzoni and Marin (2004) show that investors exhibit an asymmetric response to the degree of funding firms' projected benefit obligations due to the fact that investors paid more attention to the pension benefit accruals reported on the income statement rather than to the market values of pension assets/liabilities reported in the footnotes to the financial statements. Their cross-sectional and time-series results show that investors tend to be negatively surprised when the information regarding to the level of pension fund underfunding becomes revealed to the public. No symmetric positive response is found for overfunded plans).

Changes in pension fund tax and accounting coupled with continuing bear market since the early 2000s and low interest rates made investors aware of the challenges faced by pension funds and pressured them into adopting strategies to control pension funds' volatility and risk. To accomplish this, sponsors may need to shift their focus from a separate to a joint management of pension assets/liabilities. A joint management of pension assets/liabilities may involve strategies focusing on being able to better match pension liabilities outstanding such as duration-based immunization strategies, duration and convexity-based immunization strategies, and/or customized asset-allocation strategies.

MOTIVATION BEHIND FUZZY LIABILITY-DRIVEN PENSION FUND MANAGEMENT AND DATA DESCRIPTION

The strong equity market and the high interest rate environment over the 1990s, tax incentives, and incentives resulting from the lack of the proper regulatory/accounting base created an appearance of a sustainable pension fund environment from the point of view of regulators, sponsors, shareholders, and retirees. Regulators viewed future pension benefits of retirees secured at the level of past contributions and given the profitability/liquidity of the PBGC's pension fund assets. Sponsors enjoyed lower pension fund contributions and contribution holidays over the period of an abnormal pension fund assets' performance. Sponsors also benefited from earning greater pre-tax returns on pension fund assets and from their ability to accumulate contingency funds (Bodie (1990) states that pension fund assets may be used as a tax-sheltered contingency device, allowing sponsors to utilize pension funds to either terminate in case of bankruptcy or improve its deteriorated financial position in case of unforeseen events). Shareholders benefited from higher earnings reported on firms' income statements and greater dividend payouts, partly because of firms' stock being overpriced due to managers' incentive to manipulate earnings. Retirees enjoyed lower pension fund contributions, ad hoc benefit increases, and broader pension fund asset base.

Once the US pension crisis unfolded, it became evident that funding future pension obligations is unsustainable due to a wider discrepancy between market values of pension assets and liabilities. With this realization, regulators felt the need to change the laws so that adopted pension fund strategies will tend to be more proactive rather than reactive to pension risks involved.

Clark and Monk (2006) deem the interest of plan sponsors in defined benefits plans to re-emerge, especially in the presence of continual skilled-labour shortages, if the following changes are made: (i) the government's removes incentives for plan sponsors to take riskier pension investment strategies, provided in the form of an embedded PBGC insurance against bearing losses should the market value of pension assets fall below the market value of accrued pension liabilities; and (ii) the sponsors properly account for the unhedgeable pension fund risks (i.e., through the usage of innovative financial or insurance products).

There are many unhedgeable risks that defined benefit plans may face. First, sponsors need to be able to forecast retirees' life expectancies to determine the time period over which benefits will be collected. Clark and Monk (2006) emphasize that sponsor actuaries' underestimated retirees' life expectancies, although the underestimation can also be the result of enhanced longevity through better nutrition, healthier life styles, and/or improved standards of life. Second, sponsors may be unable to meet projected retirees' inflation-adjusted pension payments should the realized inflation rate exceed inflation rate projections. Since US DBO are rarely indexed for unpredictable CPI changes, the unhedgeable inflation risks are less severe for US pension plans. In the UK, on the contrary, all pension benefits are automatically indexed for inflation (Bodie, 1990). Third, sponsors may bear the risks related to wage inflation. The effects of rising future wages may be mitigated through either the stabilization of pension contributions or the usage of the benefit formula adjusted for future wage increases. Fourth, sponsors may face difficulties in attempting to predict future real rates over the years of employees' service up to retirement with any degree of certainty. Fifth, the reported market pension liability payment estimates depend on the spot yield curve applied to discount projected pension cash flows. This means that the market value of pension liabilities and their duration change with shifts in the discount rate term structure applied to value liabilities.

Besides all of the above unhedgeable risks that make the precise and up-to-date quantification of pension benefit assets/liabilities and other quantities of interest difficult through the usage of crisp parameters in deterministic formulas, pension fund managers should also be able to incorporate the following vagueness into their pension investment strategies. First, pension managers make subjective decisions analysing objective data and these decisions may be suboptimal due to the inherent uncertainty of financial markets. Managers may be unable to forecast market movements in the variables of interest such as stock indices, interest rates, exchange rates, etc. Second, managers make subjective decisions based on pension board's subjective assessment of the volatility of plan assets and plan sponsors/ members' risk tolerances. Third, markets do not behave in accordance with predictions of standard financial models when the market volatility is high. During these doom times, it is difficult, if not impossible, to predict the outcomes of certain events or the possibilities of major events taking place such as financial catastrophes/crises, natural calamities. Fourth, managers' decisions may be imperfect due to the unavailability (Fama (1991) suggest that the marginal benefits of acting on information should not exceed the marginal costs) or incompleteness of relevant information.

All of these unhedgeable risks as well as regulatory/accounting changes led pension managers to re-think the appropriateness of managing pension assets based on the target return on assets' (ROA) assumption (i.e., with target ROA for the year equal to the break-even rate required to meet forecasted pension benefit payments due) with an idea of either meeting or exceeding that benchmark. This pension –asset only management strategy might create significant mismatches between pension assets and pension liabilities if the realized ROA deviates from its expected level. These mismatches are problematic due to the greater pension funds' funding risk, which is a function of the current funding ratios of plan assets to plan liabilities, the ability to close funding gaps over time, the magnitude of pension contributions, the generosity of pension benefit increases, and the current pension fund portfolio allocation decisions. The need to maintain a 100% funding targets, a new requirement imposed on pension funds after PPA (2006), created incentives for pension funds to be fully-funded. To achieve full funding levels, pension funds became interested in pursuing a range of different strategies, primarily focused on strategies with a superior replication potential of pension asset cash inflows to pension liability cash outflows (i.e., cash flow matching, dedication, portfolio immunization).

We use fuzzy theory framework to be able to account for all of the above mentioned risks/ uncertainties involved in pension fund management and decision-making. We suggest that the best pension fund strategy is that of a contingent immunization. This will involve an active management of pension plan's assets until the value for the pension threshold (i.e., the fuzzy guaranteed pension benefit payments owed) is reached, at which time the pension fund manager should start pursuing an immunization strategy. This means that to be able to pursue an active pension fund investment strategy, the pension manager needs to overfund the pension plan by raising periodic pension contributions. We argue that it is difficult to achieve a true immunization strategy using crisp parameters and formulas. First, pension liabilities are neither fixed nor identifiable. Second, it is difficult to earn rates of return on the immunized portfolio that would exceed the corporate yield curve rates applied to discount pension liabilities. Many corporate bonds have embedded call and credit risks that need to be taken into consideration in projecting their holding period returns. Therefore, a fuzzy immunization strategy might be more appropriate. We include corporate defaultable bonds devoid of all other embedded options into our fuzzy pension asset portfolio. Even though our immunized portfolio is not completely riskless (One would be able to lock in a riskless rate of return on pension fund assets only by investing in a portfolio of government bonds, although this riskless rate of return would be significantly less than the liability discount rate. Then, the pension manager would need to use internal funds to pay for pension management fees and for the difference between the pension asset return and the pension liability discount rate), the usage of fuzzy duration and convexity-matched immunization, by equating the fuzzy present value of pension assets with fuzzy pension liabilities outstanding, mitigates this problem.

This study uses 18 bullet-type noncallable bonds denominated in Canadian dollars issued by several Canadian companies. For each bond, we download the cusip, coupon, maturity / announcement dates, and the S&P bond ratings from the Bloomberg database. Additionally, we download historical monthly closing bond prices and bid /ask yield quotes. For details of considered bonds, (See Table 1). We report bond coupon rates, maturity dates, S&P bond ratings, remaining times to maturity, and triangular fuzzy market prices of these bonds as of April30th, 2007 (Both, active and passive, pension fund strategies are assumed to be taken on April 30, 2007). T.F.N.s are also used to represent each credit-spread adjusted time-specific Government of Canada zero-coupon yield to discount time-specific future benefit payments. The size and timing of cash flows received by the bondholder on bullet defaultable bond holdings may change with changes in the zero-coupon appropriate term-structure interest rates and default probabilities, resulting in changes in the no-arbitrage triangular fuzzy fair bond values. For bullet-type bonds, such variables as coupon rates and maturity dates are fixed at issue (See Table 1).

The fuzzy no-arbitrage bond value is its intrinsic value on a specific date assuming that each expected bond cash flow is discounted at the fuzzy credit-spread adjusted zero-coupon interest rate with maturity equal to the maturity of that cash flow. As part of an active pension fund management strategy for surplus pension fund assets, we compare each considered bond's calculated triangular fuzzy fair (intrinsic) value to its triangular fuzzy quoted market price and make a decision with respect to which bonds to buy/sell. An active pension fund manager would purchase underpriced, relative to the calculated fair value, bonds and sell overpriced bonds.

To discount the future cash flow streams for each bond in our set, we use Government of Canada daily spot rate term structure of interest rates, extracted from the pricing data for Government of Canada bonds and treasury bills and downloaded from the Bank of Canada website. These government spot term structure rates are adjusted for the embedded default premium expected by investors for holding corporate defaultable bonds in a pension fund portfolio. Utilizing spot rate term structure, we eliminate an assumption of a flat yield curve – a known deficiency and a cornerstone of a crisp bond analysis. However, it still shifts in a parallel fashion. Each time-specific and credit-spread adjusted spot rate has three fuzzy components: the minimum, the middle, and the maximum projected yields to maturity (i.e., the least likely, the most likely, and the maximum possible fuzzy values).

As part of passive pension fund management strategy, we immunize pension liabilities of the same Canadian companies to changes in interest rates using fuzzy duration-convexity approach. This will allow pension assets to move in-sync with movements in pension liabilities irrespective of changes in the term structure of interest rates, allowing the pension manager to reach the goal of meeting projected pension payments with certainty. A fuzzy immunization involves applying fuzzy duration or fuzzy duration-convexity adjusted weights to estimated fuzzy present values of pension liabilities outstanding on the chosen immunization date of the 30th of April, 2007 to find fuzzy dollar bond investment amounts into each bond included into the bond portfolio. We approximate fuzzy pension liabilities for each firm using historical quarterly estimates of short-term and long-term debt since the year 2002. Historical quarterly firm total debt information for 11 considered Canadian companies is downloaded from the Bloomberg database. In calculating the present value of PBO we assume that the pension liability structure is known for a specific number of quarters, with the certain percentage of pension liabilities due at the end of the 1st quarter.

Assuming a known pension liability structure allows estimating the present value of projected benefit payments and matching cash inflows from investing into the pension fund portfolio with the promised stream of cash outflows (pension liabilities).

Duration is a measure of bond price sensitivity to changes in interest rates and is an important measure of price volatility from a point of view of both, active and passive (immunization), pension fund strategies. In crisp analysis, the bond's duration formula is given by formula (A.7) in Appendix 1 (Appendix 1 discusses the arbitrage-free bond valuation and the application of duration-convexity approach in crisp and fuzzy analysis).

It is the weighted average time to maturity for the promised cash flow stream, at which the bond price risk is equivalent to the coupon reinvestment risk. When the latter risks offset each other, capital gains (losses) offset coupon reinvestment losses (gains). We use the fuzzy representation of a crisp Fisher-Weil duration that relaxes the flat yield curve assumption of a Macaulay duration. The crisp Fisher-Weil duration and the immunization analysis based on Fisher-Weil duration asset-liability matching embed an unrealistic assumption of an additive spot rate interest rate process. In reality, the movements in the yield curve may be unexpected/unparallel and bond portfolio managers may have only vague/inaccurate predictions for the possible yield curve scenarios, based on their subjective opinion (Based on a set of available information) with regard to possible market movements. From the pension asset – only management perspective, the idea is to choose high-duration, low coupon rate bonds if you expect an inverted yield curve and short-duration, high coupon rate bonds if you expect a rising yield curve (Karsak, 1998). The latter perspective may be more appropriate for a speculator. From the pension asset – liability management (liability-driven investment) perspective, the idea is to immunize the portfolio of pension assets to be able to meet projected cash disbursements.

The disadvantage of creating and managing the duration-matched but not convexity-matched portfolio of pension assets versus pension liabilities are possible pension assets/liability mismatches due to the unparallel yield curve shifts or large interest rate changes. The duration approximation to the actual percentage bond price changes works well only if incremental interest rate changes are small and if the immunized portfolio is frequently re-balanced to account for changes in duration versus changes in market interest rates. To be able to match the yield curve positioning of pension fund payments, their convexity should be greater or equal to the convexity of liabilities as well. Greater convexity of assets versus liabilities at high (low) market interest rates implies shorter (longer) duration of pension assets versus the duration of pension liabilities. Shorter (longer) duration of pension assets implies a lower (greater) decline (increase) in the market value of assets versus a(n) decline (increase) in liabilities.

Because neither bond market prices nor the shape of the term structure are known precisely, we model Fisher-Weil duration/convexity as T.F.N.s. Thus, fuzzy logic provides an excellent framework for analysis in the context of choosing an optimal pension portfolio providing greater than riskless returns and, at the same time, minimizing the possibility of cash inflow – outflow mismatches.

For the purposes of conducting the fuzzy pension fund immunization strategy, we find the triangular fuzzy Fisher-Weil duration/convexity estimates for each firm's cash outflow payments. The liability payments for each quarter are assumed to be fuzzy and different, with liability projections made at the quarterly liability growth rate equal to the historical (ex-post) quarterly geometric average growth rate. The discount rates are fuzzy and different too, as was shown before: $R_j^0=(k_j^0 (\alpha), s_j^0 (\alpha))$. For each of 11 firms, we find fuzzy present value of pension liabilities (FPVL) using equation (A.16), exact (given by equation (A.12)) and approximate (given by equations (A.13) and (A.14)) fuzzy durations/convexities of projected benefit payments.

For the purpose of a pension fund asset portfolio formation, we consider the same 18 bonds. For each firm, we choose the bonds within the longest maturity spectrum to match with the long-dated nature of pension liabilities. In accordance with the latter criterion, we choose three highest duration bonds to minimize the risk of rising pension liability values due to a yield curve inversion. Then, for a successful immunization strategy, we have to set the duration/convexity of pension cash outflows equal to the weighted average of durations/convexities for corporate bonds included in the portfolio. Solving the following system of simultaneous equations, we find the weights of our investment into each bond: X_i, X_j, and X_k:

$$\begin{cases} D(L) = X_i \times D_i + X_j \times D_j + X_k \times D_k \\ C(L) = X_i \times C_i + X_j \times C_j + X_k \times C_k \end{cases}$$

In the above formula, D_L, D_i, D_j, and D_k stands for the duration of pension liability payments and the durations of three bonds: i, j, and k respectively. Similarly, C_L, C_i, C_j, and C_k stands for the convexity of pension liability payments and the convexities of three bonds: i, j, and k respectively. X_i, X_j, and X_k are the investment weights. To find out how much (in terms of the fuzzy dollar amount) we should invest in each bond, we multiply each investment weight by the FPVL for the firm (we use the AONs (AFPVL) in this study). Because we assume that the guaranteed pension liabilities are fully funded and undertake a duration and convexity-matched portfolio to the pension liability structure, with an expected yield on the portfolio equal to the liability growth rate, the future pension out-of-pocket funding should be zero or minimal.

FUZZY LIABILITY-DRIVEN INVESTMENT APPROACH TO PENSION PLAN MANAGEMENT AND RESULTS

The pension plan contributions over the life of the pension fund represent the assets of the fund which are invested in traded assets and valued at their market values. The pension liabilities represent future discounted benefit payments. The risk in pension funding is that the market value of pension assets falls short of the market value of pension liabilities, which might happen as the result of movements in the market value of pension assets/liabilities, changes in the discount or contribution rates. The scenario of a pension deficit is suboptimal from the point of view of plan members as well as sponsor firm shareholders, whose share values might be negatively affected by the poor earning prospects and lower expected firm free cash flows.

An inefficient market calls for active pension fund management strategies applied to surplus pension assets that focus on achieving a superior portfolio selection or better market timing. An efficient market calls for passive pension fund management strategies such as benchmarking pension fund returns versus bond-index fund returns are more appropriate. An alternative passive strategy is to immunize the portfolio of pension liabilities to yield curve movements by holding a pension asset portfolio with the same yield curve sensitivity. This study uses fuzzy logic to immunize pension liabilities outstanding, which allows incorporating different kinds of imprecision to make the choice between bond investment alternatives to match with projected future pension outflow payments. We discuss each strategy in turn below.

Active Pension Fund Management under Fuzzy Environment and Fuzzy Duration Analysis

If the pension fund builds excess pension assets in its plan, then an active pension fund strategy may be pursued. In this study, we compare triangular fuzzy bond intrinsic values to their triangular fuzzy market prices as of April 30th, 2007, for 18 bullet-type defaultable and noncallable Canadian bonds. We suggest that the pension fund manager should utilize the mispricing between by shorting overpriced, relative to fair value, bonds and long underpriced bonds.

As an example, we are going to go over the calculation to find the FIVB for Brookfield with a coupon rate of 8.125%. We consider the case, when all of the cash flows are crisp and equal (the last expected to be accrued cash flow is coupon income plus the face value of the bond), whereas the rates itself are fuzzy and different. Then, plugging the appropriate values in (A.6), for 0 ≤ α ≤ 1, we obtain:

$$FIVB(\alpha)=\left[\frac{8.125}{[1.11493-0.05830\alpha]^{0.13}},\frac{8.125}{[0.05830\alpha-1.00167]^{0.13}}\right](+)\left[\frac{8.125}{[1.11517-0.05830\alpha]^{0.63}},\frac{8.125}{[0.08299\alpha+1.05830]^{0.63}}\right]$$

$$(+)\left[\frac{8.125}{[1.11496-0.05830\alpha]^{1.13}},\frac{8.125}{[0.05830\alpha+1.00164]^{1.13}}\right](+)\left[\frac{108.125}{[1.11456-0.05830\alpha]^{1.63}},\frac{108.25}{[0.05830\alpha+1.00204]^{1.63}}\right]$$

The detailed numerical computations for values of 0 ≤ α ≤ 1 are omitted to save space.

After the computation for all the membership grades (α), all the monthly values were cumulated and the results show the lower and upper bounds for FIVB (α) to be 98.59 and 116.61 respectively for α=0. These bounds converge to the middle value of the FIVB of 106.95 for α=1. These values show cumulative $FIVB^d(\alpha) = (V^d(\alpha), F^d(\alpha))$.The fuzzy calculated intrinsic values for other bonds are shown in Table 2. In the same table, we report these bonds' triangular fuzzy durations/convexities (See Table 2).

To calculate the fuzzy duration(convexity) for the 8.125% coupon Brookfield bond, we apply the formula given in (A.12) and get the lower and upper bounds for FD(FC) (α) to be 1.5125 (1.963) and 1.5255 (1.982) respectively for α= 0. These bounds converge to the FD(FC) middle value of 1.5190 (1.972) for α=1. Fuzzy durations/convexities for all the other bonds are shown in Table 2 (See Table 2).

Besides fuzzy intrinsic bond values and their fuzzy durations/convexities, the table contains associated ordinary intrinsic bond values (AOIBV), and associated ordinary duration/convexity values (denoted as AOD and AOC respectively). We use the Kaufmann and Gupta's ranking procedure to calculate associate ordinary estimates (Kaufmann and Gupta's method of ranking fuzzy numbers is based on the AON calculation in accordance with the following equation: $AFN(FN(a,b,c))=\frac{(a+2b+c)}{4}$). Thus, for 8.125 coupon Brookfield bond, we substitute the calculated fuzzy upper and lower bounds together with the FIVB's middle value into an equation for AFN calculation and find an AOIVB of 107.28. Similarly, the calculated AOD equals 1.5190 (FD ranges from 1.5125 to 1.5255) and the calculated AOC equals 1.972 (FC ranges from 1.963 to 1.1982). Thus, fuzzy framework, which incorporates different kinds of imprecision into analysis, provides a nice and elegant way to get the estimated range of values for parameters of interest, crisp equivalents of which are not easily quantifiable.

After quantifying exact and approximate values for fuzzy bond values, durations, and convexities, it is important to examine whether the approximations to FIVB, FD, and FC (the so-called approximate AFIVB, AFD, and AFC) are close enough to their exact representations. This is done by studying the deviations between exact and approximate forms of fuzzy operations. If the divergence between exact and approximate forms of FIVB, FD, and FC is small and insignificant, then the analysis can be simplified. The simplification is due to the fact that fuzzy numbers are much easier to manipulate in their approximate form rather than in exact.

We calculate the left and right divergence between FIVB and AFIVB, FD and AFD, and FC and AFC at each degree of membership, α. If we represent the approximate form of a fuzzy number as $AFN=(AFN_l, AFN_m, AFN_h)$, then the calculation of left (d_l) and right (d_r) divergences is straightforward and given by the formulas below:

$$d_l = AFN_l + (AFN_m - AFN_l)\alpha - FN^{l(\alpha)}$$

$$d_h = AFN_h + (AFN_m - AFN_h)\alpha - FN^{r(\alpha)}$$

Using the above formulas, we calculate the left and right divergence for FIVB, FD, and FC, which in their approximate form are represented as:

$$AIVB = (AIVB_l, AIVB_m, AIVB_h),\ AFD = (AFD_l, AFD_m, AFD_h)\ and\ AFC = (AFC_l, AFC_m, AFC_h).$$

The deviations between FIVB and AFIVB, FD and AFD, and FC and AFC for 8.125% coupon Brookfield bond are depicted in Appendix 2. We find that the maximum deviation on the left of FIVB is at α=0.5 with a value of 0.1779, and the maximum deviation on the right of FIVB is at α=0.5 with a value of 0.1458. For this bond's fuzzy duration and convexity, we find that the exact and approximate forms of fuzzy representation converge. The left and right maximum percentages of deviations between FN and AFN (The measures given by the formulas below represent the calculations in percentage terms of the FN deviation from the width of left and right forms of AFN. The left and right percentage deviations are calculated for fuzzy intrinsic bond values, their durations and convexities) in question, $d_{l(\%)}$ and $d_{r(\%)}$, are calculated in accordance with the formulas below:

$$d_{l(\%)} = 100\% \times \frac{d_l}{(AFN_m - AFN_l)}$$

$$d_{r(\%)} = 100\% \times \frac{d_r}{(AFN_h - AFN_m)}$$

For the 8.125% coupon Brookfield bond, the computed percentage divergence between FIVB estimates in exact and approximate representations is insignificant and amounts to 1.1534% and 1.0919% respectively. The computed percentage divergences for fuzzy exact and approximate duration/convexity measures converge. In fact, for most considered bonds, the latter estimates converge and only estimates for bonds with divergent duration/convexity measures are reported. We present left and right deviations as well as the left and right maximum percentages of deviations between FIVB and AFIVB (between FD and AFD, and between FC and AFC) (See Table 3 and Table 4).

Observing results reported in tables 3 and 4, we note that most of the reported percentage deviations are less than 1%, implying that the AFIVB, AFD, and AFC are reliable enough representations of FIVB, FD, and FC respectively. Therefore, we use approximate forms of these values in calculations below.

We select an appropriate active pension fund management strategy by comparing the fuzzy current market price (AOCMP) of each bond to its associate ordinary fuzzy intrinsic value equivalent (the bond's fair value; AOIVB). Active portfolio management techniques are appropriate in an inefficient bond market for pension managers who wish to earn abnormal returns on assets that are mispriced relative to their fair values. Active managers' short bonds, overpriced relative to their fair value, and long underpriced bonds.

We employ the Kaufmann and Gupta's fuzzy ranking procedure to make our investment decision. For each considered bond, Table 5 reports its AOCMP and triangular fuzzy current market price, AOIVB and triangular fuzzy intrinsic value of the bond, and states the active pension fund management decision (See Table 5).

For our example bond, the AOCMP of 104.39 falls short of the 107.28 AOIVB equivalent and the decision is to take a long position in this bond. The optimal active strategy of the pension manager with regard to the rest of considered bonds is given in table 6 below.

Passive Pension Fund Management under Fuzzy Environment

Immunization in conventional finance theory is accomplished by calculating the duration/convexity of the promised pension outflows and then investing pension assets into a bond portfolio with an identical weighted average of durations / convexities of bonds included in the portfolio. We discount projected pension outflows (expected bond cash inflows) using firm (bond) credit spread-adjusted Government of Canada spot rates applicable to the 30th of April, 2007.

In an efficient bond market, the immunization strategy is most appropriate. It locks in pension inflows to meet projected fuzzy pension outflows and is successful only if the following conditions are met: (i) the present values of pension assets and liabilities are equivalent; (ii) the duration/convexity of the bond portfolio are equal to the duration/convexity of the projected stream of pension liability payments. We find fuzzy present values of liability payments (FPVL) using equation (A.16) and appropriate time-specific, firm credit-spread adjusted Government of Canada spot rates to discount projected future pension outflows.

We choose George Weston Limited as an example company for an immunization strategy. We assume that future pension liabilities and discount rates are fuzzy and different. Plugging the appropriate fuzzy values in (A.16), for $0 \le \alpha \le 1$, we obtain:

$$FIVB(\alpha) = \left[\frac{7258.83}{[1.09814 - 0.04371\alpha]^{1.92}}, \frac{7258.83}{[1.01072 - 0.04371\alpha]^{1.92}}\right](+)\left[\frac{7248.43}{[1.09825 - 0.04371\alpha]^{1.67}}, \frac{7248.43}{[1.01083 + 0.04371\alpha]^{1.67}}\right](+)$$

$$\left[\frac{7238.05}{[1.09841 - 0.4371\alpha]^{1.42}}, \frac{7238.05}{[1.01100 + 0.04371\alpha]^{1.42}}\right](+)\left[\frac{7227.68}{[1.09863 + 0.04371\alpha]^{1.17}}, \frac{7227.68}{[1.01122 + 0.04371\alpha]^{1.17}}\right](+)$$

$$\left[\frac{7217.33}{[1.09892 + 0.04371\alpha]^{0.92}}, \frac{7217.33}{[1.01150 + 0.04371\alpha]^{0.92}}\right](+)\left[\frac{7206.99}{[1.09927 - 0.04371\alpha]^{0.67}}, \frac{7206.99}{[1.01185 + 0.04371\alpha]^{0.67}}\right](+)$$

$$\left[\frac{7067}{[1.09970 - 0.04371\alpha]^{0.41}}, \frac{7067}{[1.01229 + 0.04371\alpha]^{0.41}}\right](+)\left[\frac{7158}{[1.10016 - 0.04371\alpha]^{0.16}}, \frac{7158}{[1.01275 + 0.04371\alpha]^{0.16}}\right]$$

We omit all details of the above calculations for the rest of the firm sample to save space. FPVL, FD, and FC as well as their approximate representations for each considered firm pension liabilities are given in Table 6 (See Table 6).

The AOPVL, AOD, and AOC for George Weston are \$54,324.66, 1.12, and 1.29 respectively. The lower and the upper bounds for FPVL (at α=0) are \$52,215.22 and \$56,865.55 respectively, converging to the middle value of \$54,427.30 (at α=1). The AOPVL equals to \$54,483.85. The AOD is 1.0303, whereas the FD ranges from 1.0168 to 1.0441. Likewise, AOC is 0.9650 versus the estimated range for the FC, ranging from 0.9465 to 0.9838. We report left and right FPVL and AFPVL (FD and AFD, FC and AFC) maximum deviations and percentage deviations for each firm pension liabilities (See Table 7 and Table 8).

The next question is whether there is a divergence or convergence in exact and approximate representations of fuzzy liability payments, their durations, and convexities? If left and right percentages of deviations, at each degree of membership, α, are small, then there is no reason use exact fuzzy numbers in further calculations.

The deviations between FPVL and AFPVL, between FD and AFD, and between FC and AFC are depicted in Appendix 3. For the example company, the left maximum and percentage deviations from FPVL are \$30.34 and 1.24% (at α=0.5), whereas the right maximum and percentage deviations from FPVL are \$26.30 and 0.88% (at α=0.5) respectively. Observing results in Table 8, we find that percentage deviations between FPVL and AFPVL for all considered firms is less than 1% on the right and not much higher than 1% on the left (the highest deviation is 4.42%), allowing us to conclude that AFPVL is a reliable estimate of FPVL. The maximum left and right percentage deviations from FD are 0.5314% and 0.5184% (at α=0.5) for the example company, whereas the maximum left and right percentage deviations from FC are 0.5802% and 0.5654% (at α=0.5) respectively. The percentage deviations between the FD and AFD, and between FC and AFC for the rest of considered firms are less than 1%. Therefore, the usage of AFD and AFC in place of their exact counterparts is reliable and simplifies further computational work (Because it is often difficult to obtain precise or precise within a small range of possible values estimates of expected parameters of interest, other ways of approximating fuzzy parameters may be called for. However, the limited time span did not allow us to think about other more close approximations to the lower and upper bounds of fuzzy variables for $0 \leq \alpha \leq 1$. Consequently, this issue is left for further research).

In Table 9, we group considered for the immunization strategy bonds into the short-term (0-5 years), medium-term (5-10 years), and long-term (>10 years) maturity buckets and rank them using their duration/convexity removals in accordance with the Kaufmann and Gupta's ranking criterion. Duration/convexity based removals are reported in columns 5 and 6 of Table 9. None of the calculated removals are the same (i.e., no fuzzy class can be formed), meaning that the application of the removal criterion is sufficient for ranking purposes. Reported results show that the linear order of T.F.N.'s for the short-maturity bucket is: A1<A2<A3<A4<A5, the linear order of T.F.N.'s for the medium-maturity bucket is: A6<A7<A8<A9<A10<A11<A12, and the linear order of T.F.N.'s for the long-maturity bucket is: A13<A14<A15<A16<A17. Based on calculated removals, bonds ranked as A_{15}, A_{16}, and A_{17} should be included in the optimal pension fund portfolio (See Table 9).

The AD and AC for projected George Weston's liability payments are estimated to be 1.03 and 0.48 respectively. We select bonds for the pension asset portfolio using the longest maturity bonds at our disposal to match with the long-dated nature of pension liabilities. Applying this selection criterion to the sample of 18 bullet-type available bonds, we take the 23-year 7.63%-coupon and the 36-year 6.71%-coupon Canadian Rail bonds, and the 31-year 11.4% coupon Loblaw bond to form the portfolio (In alternative calculations, we choose one longest maturity bond from each maturity bucket, the short-term (0-5 years), medium-term (6 – 10 years), and long-term (>10 years), resulting in the portfolio investment strategy into the 10-year 5.53% coupon Canadian Tire bond, the 14-year 6.0% coupon Loblaw bond, and the 36-year 6.71% Canadian Rail bond. Results of this immunization strategy are reported in Table 11 (See Table 11)). AFD values of portfolio bonds are 10.0255, 14.6797 and 11.6348 respectively, whereas their AFC are 71.86, 167.01 and 107.21 respectively. To find the weights of our investment in the above bonds, we solve a system of the following simultaneous equations using associate ordinary duration/convexity parameters for pension assets and liabilities. We set the associate ordinary duration and associative ordinary convexity of pension outflow requirements equal to the weighted average of durations/convexities of bonds included in the portfolio below:

$$\begin{cases} 1.03 = 10.03X_{23} + 11.63X_{31} + 14.68X_{36} \\ 0.48 = 71.86X_{23} + 107.21X_{31} + 167.01X_{36} \end{cases},$$

where X_{23}, X_{31}, and X_{36} denote investment weights. Solving the system, we find that that the optimal strategy is to short 2705.50% of the 23-year bond, long the 4583.63% of the 31-year bond, and short 1778.13% of the 36-year bond. For an immunization strategy to work, we need to invest in the above bonds an amount equal to the fuzzy present value of the pension liability payments. The FPVL for the example firm is ($52,215.22, $54,427.30, $56,865.55) with an AOPVL equal to $54,483.85. This strategy ensures that the fuzzy dollar present value investment of shorting $1,474,060.81 and $968,792.45 worth of 23-year and 36-year bonds respectively and investing $2,497,337.10 worth into the 31-year bond equals to the fuzzy dollar present value of liability payments. The rest of immunization results are presented in Table 10. The fuzzy immunization strategy works to ensure that pension assets will be sufficient to cover pension liabilities irrespective of changes in the yield curve (See Table 10).

The disadvantage of the taken fuzzy liability-driven approach to the management of pension assets is due to observable significant deviations of some fuzzy parameters from their associated ordinary equivalents. However, the fuzzy approach does offer a superior approach for the liability-driven investment because it is often difficult to obtain precise estimates for the parameters of interest with any degree of certainty.

CONCLUSION

Pension benefits are a function of market interest rates, the time the benefits are due, the credit risk for pension plan members, and the liquidity of pension assets. Precise quantification of these inputs is rarely available. In part, this is because precise formulas cannot be used to capture inherently imprecise processes and an exact assessment of inputs is not possible. We use triangular fuzzy numbers to characterize these imprecise inputs to effectively tackle the ambiguities involved in the crisp analysis of pension fund management.

We apply the fuzzy decision-theory framework to the problem of the liability-driven approach for the management of pension funds. Within the context of the asset-liability management, the best approach to ensure the return on pension assets equals or exceeds the rate of growth in pension benefit payments is to immunize against changes in pension fund worth due to interest rate movements. Within the context of the asset-only management, the best approach is to try to achieve a certain return on pension assets without an appropriate consideration given to pension benefit payments. Considering recent regulatory/accounting changes, poor equity market performance and low market yields, pension funds may find that the strategy of a contingent immunization is the most optimal. The latter strategy combines the asset-liability management of funds minimally necessary to meet guaranteed projected benefit obligations and the asset-only management of surplus pension funds. We apply the contingent immunization strategy to the Canadian firm-level data, report results, and conclude that fuzzy methods are more effective than conventional crisp methods of pension fund management.

REFERENCES

Anonymous, 2006, OECD guidelines on pension fund asset management, Recommendation of the Council, Directorate for financial and enterprise affairs.

Ambachtsheer, K. P. (2002). A beautiful plan. *Benefits Canada* 26 (10), p.19

Bodie, Z. (1990). The ABO, the PBO, and pension investment policy. *Financial Analyst Journal*.

Fama, E. F. (1991). Efficient capital markets: II, *The Journal of Finance* 21, pp. 1575-1617.

Franzoni, F., Marin J. M. (2004). Pension plan funding and stock market efficiency, Working paper, HEC School of Management.

Clark, G., Monk A. H. B. (2006). The "crisis" in defined benefit corporate pension liabilities: current solutions and future prospects, Working paper, University of Oxford.

Cooke, M. (2003). The Canada pension plan goes to market, *Canadian Review of Social Policy* 51, p. 126-131.

Karsak, E. E. (1998). Measures of liquidity risk supplementing fuzzy discounted cash flow analysis, *The Engineering Economist* 43, pp. 331-344.

Kopcke, R. (2006). Managing the risk in pension plans and recent pension reforms, Working paper, Federal Reserve Bank of Boston.

Inkmann, J., Blake D. (2007). Pension liability valuation and asset allocation in the presence of funding risk, Working paper, Tilburg University.

MacDonald, J. (2007). Experts track continuing evolution of US pension system, *Employee benefit research institute* 28, p.8.

Rauh, J.D. (2006). Risk shifting versus risk management: investment policy in corporate pension plans, Working paper, University of Chicago.

TABLE 1
SUMMARY CHARACTERISTICS OF CANADIAN CORPORATE BULLET-TYPE NONCALLABLE BONDS

We report bond coupon rates, maturity dates, S&P bond ratings, remaining times to maturity, and triangular fuzzy market prices of these bonds as of April 30th, 2007.

Name of the company	Coupon Rate	Maturity	Rating	Number of years to maturity	Fuzzy estimated current market price
Enbridge	6.09	20080305	A-	1	(77.3451,101.2665,125.1879)
Encana	3.6	20080915	BBB+	2	(95.7878,98.8459,101.9040)
Brookfield	8.125	20081215	BBB	2	(70.5373,104.3935,138.2497)
Talisman	8.06	20090916	BBB	3	(70.94035, 107.761, 144.5817)
Can Tire	5.53	20100510	BBB+	4	(84.94868, 102.839, 120.7293)
Talisman	4.44	20110127	BBB	4	(91.37664, 99.342, 107.3074)
Brookfield	7.125	20120615	BBB	6	(75.39844, 107.3215, 139.2446)
Telus	4.5	20120315	BBB+	5	(82.34972, 97.1489, 111.9481)
Encana	4.3	20120312	BBB+	5	(90.4075,98.9406,107.4737)
Telus	5	20130306	BBB+	7	(76.64673, 97.6233, 118.5999)
Enbridge	4.67	20130325	A-	6	(87.0775,100.0360,112.9945)
Loblaw	6	20140303	BBB	7	(77.07928, 106.9396, 136.7999)
Shaw	5.7	20170302	BB	10	(73.82414, 99.3355, 124.8469)
PetroCanada	9.25	20211015	BBB	15	(73.20935, 126.741, 180.2727)
CNRail	7.625	20230515	A-	17	(83.42173, 117.517, 151.6123)
Weston	12.7	20301108	BBB	24	(98.79652, 188.544, 278.2915)
Loblaw	11.4	20310523	BBB	25	(89.21625, 177.141, 265.0658)
CNRail	6.712	20360715	A-	30	(63.58683, 108.375, 153.1632)

TABLE 2
REPORTED EXACT AND APPROXIMATE REPRESENTATIONS OF FUZZY FAIR BOND VALUES AND THEIR FUZZY DURATIONS/CONVEXITIES

This table reports fuzzy intrinsic bond values (FIVB), and fuzzy durations/convexities (FD/FC) for each bond in our set. FIVB are calculated by discounting future cash flow streams for each bond using time-specific, credit-spread adjusted Government of Canada daily spot rates. The table also contains associated ordinary intrinsic bond values (AOIBV), and associated ordinary duration/convexity values (AOD/AOC), calculated using the Kaufmann and Gupta's ranking procedure.

Company (coupon rate)	Estimated approximate FIVB	Estimated AOIVB	Estimated approximate FD	Estimated AOD	Estimated approximate convexity	Estimated AOC
Enbridge (6.09 %)	(98.20,101.72,105.72)	101.79	(0.8344,0.8347,0.8350)	0.8347	(0.7690,0.7693,0.7696)	0.7693
Encana (3.6%)	(94.13,97.84,101.82)	97.91	(1.3534,1.3540,1.3547)	1.354	(1.6037,1.6046,1.6055)	1.6046
Brookfield (8.13%)	(98.59,106.95,116.61)	107.28	(1.5125,1.5190,1.5255)	1.519	(1.963,1.972,1.982)	1.972
Talisman (8.06)	(95.68,106.57,119.43)	107.06	(2.1912,2.2034,2.2154)	2.2033	(3.625,3.649,3.674)	3.649
Can Tire (5.53%)	(94.10,102.75,112,57)	103.04	(2.7433,2.7604,2.7770)	2.7603	(5.411,5.450,5.488)	5.45
Talisman (4.44%)	(90.11,97.68,106.12	97.9	(3.4343,3.4504,3.4660)	3.4503	(7.966,8.011,8.055)	8.011
Brookfield (7.13%)	(93.75,112.50,136.70)	113.86	(4.2260,4.3261,4.4200)	4.3246	(12.28,12.63,12.97)	12.63
Telus (4.5%)	(83.29,96.42,112.37)	97.12	(4.3541,4.4003,4.4438)	4.3996	(12.38,12.54,12.70)	12.54
Encana (4.3%)	(87.70,95.59,104.43)	95.83	(4.3834,4.4092,4.4341)	4.409	(12.47,12.56,12.65)	12.56
Telus (5.0%)	(81.67,99.49,122.73)	100.84	(5.0914,5.2134,5.3259)	5.211	(17.19,17.70,18.17)	17.69
Enbridge (4.67%)	(83.29,94.58,107.82)	95.07	(5.3878,5.4202,5.4512)	5.42	(17.92,18.07,18.21)	18.06
Loblaw (6.0%)	(82.37,103.73,132.89)	105.68	(5.5073,5.6837,5.8460)	5.6802	(20.19,21.01,21.78)	21
Shaw (5.7%)	(76.55,98.52,129.55)	100.79	(7.1996,7.5703,7.9125)	7.5632	(35.22,37.58,39.79)	37.56
PetroCanada (9.25%)	(101.64,136.37,190.55)	141.23	(8.2619,9.1327,9.9882)	9.1289	(51.29,58.93,66.61)	58.94
CNRail (7.63%)	(102.45,131.60,174.22)	134.97	(9.1513,10.0302,10.8901)	10.0255	(63.50,71.86,80.22)	71.86
Weston (12.7%)	(132.47,198.41,336.70)	216.5	(8.8181,11.1762,13.8688)	11.2598	(69.86,99.19,134.87)	100.78
Loblaw (11.4%)	(117.27,181,36,324,84)	201.21	(8.9183,11.5374,14.5460)	11.6348	(71.90,105.30,146.35)	107.21
CNRail (6.71%)	(86.67,128.07,210.35)	138.29	(11.7238,14.6067,17.7817	14.6797	(119.53,165.18,218.15)	167.01

TABLE 3
DIVERGENCE BETWEEN FUZZY INTRINSIC BOND VALUES AND THEIR ASSOCIATED ORDINARY EQUIVALENTS

This table reports the left and right maximum deviations and percentage deviations between the fuzzy intrinsic bond value and its associated ordinary equivalent for each bond in our set.

Name of the company (coupon rate)	a	d_l between FIVB and AFIVB	d_l(%) between FIVB and AFIVB	a	d_r between FIVB and AFIVB	d_r(%) between FIVB and AFIVB
Enbridge (6.09 %)	0.5	0.0386	1.2028	0.5	0.0342	1.1522
Encana (3.6%)	0.5	0.0346	0.6357	0.5	0.0314	0.6176
Brookfield (8.13%)	0.5	0.1779	1.1534	0.5	0.1458	1.0919
Talisman (8.06)	0.5	0.2748	0.9228	0.5	0.2209	0.8781
Can Tire (5.53%)	0.5	0.16	0.5519	0.5	0.1361	0.5345
Talisman (4.44%)	0.5	0.1176	0.3818	0.5	0.1027	0.373
Brookfield (7.13%)	0.5	0.7896	0.6704	0.5	0.5816	0.6426
Telus (4.5%)	0.5	0.3935	0.5242	0.5	0.3125	0.5069
Encana (4.3%)	0.5	0.1258	0.3029	0.5	0.1101	0.297
Telus (5.0%)	0.5	0.7865	0.5828	0.5	0.5772	0.561
Enbridge (4.67%)	0.5	0.2661	0.3542	0.5	0.2212	0.346
Loblaw (6.0%)	0.5	1.1599	0.6193	0.5	0.8082	0.4946
Shaw (5.7%)	0.5	1.368	0.4898	0.5	0.9227	0.4739
PetroCanada (9.25%)	0.5	3.0972	0.4708	0.5	1.8521	0.4565
CNRail (7.63%)	0.5	2.0694	0..3585	0.5	1.3388	0.35
Weston (12.7%)	0.5	13.4578	0.5488	0.5	5.5742	0.5312
Loblaw (11.4%)	0.5	15.2159	0.578	0.5	5.846	0.5586
CNRail (6.71%)	0.5	7.376	0.3945	0.5	3.2903	0.385

TABLE 4
DIVERGENCE BETWEEN FUZZY DURATION/COVEXITY BOND VALUES AND THEIR ASSOCIATED ORDINARY EQUIVALENTS

This table reports the left and right maximum deviations and percentage deviations between the fuzzy duration/convexity value and its associated ordinary equivalent for each bond in our set. Results for bonds exhibiting convergence between fuzzy and associate ordinary estimates of durations/convexities are omitted.

Name of the company (coupon rate)	a	d_l between FD and AFD	d_l(%) between FD and AFD	a	d_r between FD and AFD	d_r(%) between FD and AFD	a	d_l between FC and AFC	d_l(%) between FC and AFC	a	d_r between FC and AFC	d_r(%) between FC and AFC
Enbridge (6.09 %)	0.5	0	0.291	0.5	0	0.2791	0.5	0	0.291	0.5	0	0.2991
Encana (3.6%)	0.5	0	0.0549	0.5	0	0.1492	0.5	0	0.0601	0.5	0	0.0584
PetroCanada (9.25%)	1	0	0	1	0.0008	0.087	1	0	0	1	0.3567	0.0273
CNRail (7.63%)	1	0	0	1	0	0.0035	1	0	0	1	0.0183	0.2185
Weston (12.7%)	1	0.0218	0.808	0.5	0.0589	2.4971	0.5	0.5489	1.5383	0.5	0.9795	3.3398
Loblaw (11.4%)	1	0.0219	0.7263	0.5	0.0717	2.7358	1	0.6135	1.4946	0.5	1.2147	3.6361
CNRail (6.71%)	1	0.0134	0.4213	0.5	0.0578	2.006	1	0.5399	1.0193	0.5	1.2337	2.7025

TABLE 5
RESULTS OF THE ACTIVE PENSION FUND STRATEGY

This table reports an appropriate active pension fund management strategy by comparing the fuzzy current market price (AOCMP) of each bond to its associate ordinary equivalent (AOIVB). Investment decisions with respect to which bonds to long/short are made on the basis of the Kaufmann and Gupta's fuzzy ranking procedure.

Bond (coupon rate)	Fuzzy estimated Current Market Price (FCMP)-AOFCMP	Estimated approximate FIVB-AOFIVB	Comparison of AOFCMP and AOFIVB	Strategy
Enbridge (6.09%)	(77.35,101.27,125.19)-101.27	(98.20,101.72,105.72)-101.79	AOFCMP<AOFIVB	Long
Encana (3.6%)	(95.79,98.85,101.90)-98.85	(94.13,97.84,101.82)-97.91	AOFCMP>AOFIVB	Short
Brookfield (8.13%)	(70.54,104.39,138.25)-104.39	(98.59,106.95,116.61)-107.28	AOFCMP<AOFIVB	Long
Talisman (8.06%)	(70.94,107.76,144.58)-107.76	(95.68,106.57,119.43)-107.06	AOFCMP>AOFIVB	Short
Can Tire (5.53%)	(84.95,102.84,120.73)-102.84	(94.10,102.75,112,57)-103.04	AOFCMP<AOFIVB	Long
Talisman (4.44%)	(91.38,99.34,107.31)-99.34	(90.11,97.68,106.12)-97.90	AOFCMP>AOFIVB	Short
Brookfield (7.13%)	(75.40,107.32,139.24)-107.32	(93.75,112.50,136.70)-113.86	AOFCMP<AOFIVB	Long
Telus (4.5%)	(82.35,97.15,111.95)-97.15	(83.29,96.42,112.37)-97.12	AOFCMP>AOFIVB	Short
Encana (4.3%)	(90.41,98.94,107.47)-98.94	(87.70,95.59,104.43)-95.83	AOFCMP>AOFIVB	Short
Telus (5.0%)	(76.65,97.62,118.60)-97.62	(81.67,99.49,122.73)-100.84	AOFCMP<AOFIVB	Long
Enbridge (4.67%)	(87.08,100.04,112.99)-100.04	(83.29,94.58,107.82)-95.07	AOFCMP>AOFIVB	Short
Loblaw (6.0%)	(77.08,106.94,136.86)-106.94	(82.37,103.73,132.89)-100.84	AOFCMP>AOFIVB	Short
Shaw (5.7%)	(73.82, 99.34,124.85)-99.34	(76.55,98.52,129.55)-100.79	AOFCMP<AOFIVB	Long
PetroCanada (9.25%)	(73.21,126.74,180.27)-126.74	(101.64,136.37,190.55)-141.23	AOFCMP<AOFIVB	Long
CNRail (7.63%)	(83.42,117.52,151.61)-117.52	102.45,131.60,174.22) -134.97	AOFCMP<AOFIVB	Long
Weston (12.7%)	(98.80,188.54,278.29)-188.54	(132.47,198.41,336.70)-216.50	AOFCMP<AOFIVB	Long
Loblaw (11.4%)	(89.22,177.14,265.07)-177.14	(117.27,181,36,324,84)-201.21	AOFCMP<AOFIVB	Long
CNRail (6.72%)	(63.59,108.38,153.16)-108.38	(86.67,128.07,210.35)-138.29	AOFCMP<AOFIVB	Long

TABLE 6
REPORTED EXACT AND APPROXIMATE REPRESENTATIONS OF FUZZY INTRINSIC PENSION LIABILITY VALUES AND THEIR FUZZY DURATIONS/CONVEXITIES

This table reports fuzzy intrinsic liability values (FPVL), and their fuzzy durations/convexities (FD/FC) for each firm in our set. FPVL are calculated by discounting future cash flow streams for each bond using time-specific, firm credit-spread adjusted Government of Canada daily spot rates. The table also contains associated ordinary intrinsic liability values (AOPVL), and associated ordinary duration/convexity values (AOD/AOC), calculated using the Kaufmann and Gupta's ranking procedure.

Name of the firm	Estimated approximate FPVL	Liability structure is assumed to be known for (quarters)	Estimated AOPVL	Estimated approximate FD	Estimated AOD	Estimated approximate FC	Estimated AOC
Telus	(69183.06,75518.17,82995.56)	23	75803.73	(2.40,2.49,2.59)	2.49	(4.86,5.14,5.44)	5.15
CNRail	(120198.33,131461.26,144594.01)	27	131928.72	(3.15,3.26,3.37)	3.259	(7.81,8.19,8.59)	8.20
Encana	(552204.96,614188.41,686592.03)	29	616793.45	(4.37,4.46,4.56)	4.464	(12.95,13.38,13.80)	13.38
Enbridge	(121414.47,128310.47,135989.81)	13	128506.30	(1.66,1.69,1.72)	1.693	(2.30,2.36,2.42)	2.36
CanTire	(38678.31,44095.85,50798.06)	33	44417.02	(4.01,4.20,4.38)	4.198	(12.15,12.95,13.81)	12.97
Talisman	(113044.16,126142.31,141867.15)	23	126798.98	(3.00,3.10,3.20)	3.095	(6.78,7.10,7.44)	7.11
PetroCan	(99807.94,119501.50,145659.30)	43	121117.57	(5.16,5.49,5.82)	5.491	(19.74,21.60,23.56)	21.63
Brookfield	(938224.19,1108154.67,1328391.25)	23	1120731.20	(3.40,3.52,3.65)	3.524	(8.14,8.58,9.02)	8.58
Weston	(52215.22,54427.30,56865.55)	8	54483.85	(1.02,1.03,1.04)	1.030	(0.94,0.00,0.98)	0.48
Shaw	(48901.92,53488.20,58889.42)	23	53691.94	(2.57,2.67,2.75)	2.659	(5.40,5.69,6.00)	5.70
Loblaw	(140677.00,172455.20,217156.45)	43	175685.96	(4.65,5.05,5.47)	5.054	(16.91,19.09,21.47)	19.14

TABLE 7
DIVERGENCE BETWEEN FUZZY INTRINSIC LIABILITY VALUES AND THEIR ASSOCIATED ORDINARY EQUIVALENTS

This table reports the left and right maximum deviations and percentage deviations between the fuzzy intrinsic liability value and its associated ordinary equivalent for each firm in our set.

Name of the company (coupon rate)	α	d_l between FPVL and AFPVL	d_l(%) between FPVL and AFPVL	α	d_r between FPVL and AFPVL	d_r(%) between FPVL and AFPVL
Telus	0.5	158.923	2.125	0.5	127.690	0.576
CNRail	0.5	257.263	1.959	0.5	211.517	0.429
Encana	0.5	1428.767	1.973	0.5	1182.984	0.357
Enbridge	0.5	105.293	1.371	0.5	90.875	0.598
CanTire	0.5	182.575	2.724	0.5	140.215	0.471
Talisman	0.5	367.782	2.339	0.5	291.524	0.564
PetroCan	0.5	953.981	3.647	0.5	675.778	0.481
Brookfield	0.5	7349.451	3.337	0.5	5323.130	0.743
Weston	0.5	30.336	1.244	0.5	26.299	0.883
Shaw	0.5	113.073	2.093	0.5	91.357	0.547
Loblaw	0.5	1974.845	4.418	0.5	1296.891	0.601

TABLE 8
DIVERGENCE BETWEEN FUZZY DURATION/CONVEXITY LIABILITY VALUES AND THEIR ASSOCIATED ORDINARY EQUIVALENTS

This table reports the left and right maximum deviations and percentage deviations between the fuzzy duration/convexity value and its associated ordinary equivalent for each firm in our set.

Name of the company (coupon rate)	a	d_l between FD and AFD	d_l(%) between FD and AFD	a	d_r between FD and AFD	d_r(%) between FD and AFD	a	d_l between FC and AFC	d_l(%) between FC and AFC	a	d_r between FC and AFC	d_r(%) between FC and AFC
Telus	0.5	0.0007	0.6813	0.5	0.0007	0.7058	0.5	0.0026	0.8473	0.5	0.0025	0.8685
CNRail	0.5	0.0004	0.4056	0.5	0.0005	0.4390	0.5	0.0022	0.5524	0.5	0.0022	0.5835
Encana	0.9	0.0000	0.0006	0.9	0.0000	0.0303	0.5	0.0005	0.1239	0.5	0.0007	0.1549
Enbridge	0.5	0.0001	0.4061	0.5	0.0001	0.4075	0.5	0.0003	0.4789	0.5	0.0003	0.4783
Can Tire	0.5	0.0007	0.3515	0.5	0.0008	0.4292	0.5	0.0047	0.5558	0.5	0.0051	0.6300
Talisman	0.5	0.0003	0.3371	0.5	0.0004	0.3789	0.5	0.0017	0.4916	0.5	0.0017	0.5296
PetroCan	0.5	0.0011	0.3145	0.5	0.0015	0.4716	0.5	0.0118	0.6015	0.5	0.0140	0.7551
Brookfield	0.5	0.0002	0.1231	0.5	0.0003	0.2036	0.5	0.0014	0.3231	0.5	0.0017	0.3959
Weston	0.5	0.0001	0.5309	0.5	0.0001	0.5180	0.5	0.0001	0.5802	0.5	0.0001	0.5654
Shaw	0.5	0.5558	0.5549	0.5	0.0005	0.5849	0.5	0.0022	0.7110	0.5	0.0021	0.7369
Loblaw	0.5	0.0027	0.6550	0.5	0.0035	0.8822	0.5	0.0245	1.0257	0.5	0.0272	1.2490

TABLE 9
MATURITY-GROUPED CANADIAN CORPORATE DEFAULTABLE AND NONCALLABLE BONDS

This table groups, considered for the immunization strategy, bonds into the short-term (0-5 years), medium-term (5-10 years), and long-term (>10 years) maturity buckets and ranks them using their duration/convexity removals in accordance with the Kaufmann and Gupta's ranking criterion.

Panel A : Short-maturity group				
Company (coupon rate)	Years to maturity	Number	Estimated AOD	AOC
Enbridge (6.09 %)	8	A1	0.8347	0.7693
Encana (3.6%)	8	A2	1.354	1.6046
Brookfield (8.13%)	8	A3	1.519	1.972
Talisman (8.06)	9	A4	2.2033	3.649
CanTire (5.53%)	10	A5	2.7603	5.45
Panel B: Medium-maturity group				
Company (coupon rate)	Years to maturity	Number	Estimated AOD	AOC
Talisman (4.44%)	11	A6	3.4503	8.011
Brookfield (7.13%)	12	A7	4.3246	12.63
Telus (4.5%)	12	A8	4.3996	12.54
Encana (4.3%)	12	A9	4.409	12.56
Telus (5.0%)	13	A10	5.211	17.69
Enbridge (4.67%)	13	A11	5.42	18.06
Loblaw (6.0%)	14	A12	5.6802	21
Panel C: Long-maturity group				
Company (coupon rate)	Years to maturity	Number	Estimated AOD	AOC
Shaw (5.7%)	17	A13	7.5632	37.56
PetroCan (9.25%)	21	A14	9.1289	58.94
CNRail (7.63%)	23	A15	10.0255	71.86
Loblaw (11.4%)	31	A16	11.6348	107.21
CNRail (6.71%)	36	A17	14.6797	167.01

TABLE 10
RESULTS OF THE PASSIVE PENSION FUND IMMUNIZATION STRATEGY

Table 10 reports immunization results including characteristics of bonds (i.e., coupon rate, remaining time to maturity) included into the portfolio of pension fund assets, percentage investment weights, dollar investment amounts, and types of positions taken (i.e., long/short). To match with the long-dated nature of liability payments, we use the longest-maturity bonds to form the immunized portfolio. Namely, we use the 23-year 7.63%-coupon and the 36-year 6.71%-coupon Canadian Rail bonds, and the 31-year 11.4% coupon Loblaw bond. The investment weights are chosen in such a way that the weighted average duration/convexity of the portfolio equates the duration/convexity of estimated periodic liability payments. The dollar investment amounts into each bond are equal to the product of that bond's percentage investment weight and the present value of the projected pension payments for the considered firm.

Company (FPVL)-AFPVL	Bond Selected	Maturity	Investment Weight	Investment Amount	Investment Strategy
Telus	Weston	30	8342.13%	$6,323,642.32	long
(68814.62,75355.32,83090	Loblaw	31	-8968.95%	-$6,798,799.33	short
.81) - 75654.02	CNRail	36	726.83%	$550,960.74	long
CNRail (119723.64,	PetroCanada	21	221.28%	$291,928.51	long
131267.65,144750.75) -	Weston	30	16.53%	$21,813.53	long
131752.43	Loblaw	31	-137.81%	-$181,813.32	short
Encana (550096.56,	Weston	30	5056.42%	$31,187,679.35	long
613151.27, 686934.93) -	Loblaw	31	-5343.50%	-$32,958,386.23	short
615833.50	CNRail	36	387.08%	$2,387,500.33	long
Enbridge (120837.31,	Weston	30	9725.29%	$12,497,614.98	long
128071.80, 136145.56) -	Loblaw	31	-10496.27%	-$13,488,368.19	short
128281.62	CNRail	36	870.98%	$1,119,259.51	long
Can Tire (38497.71,	Weston	30	5573.76%	$2,475,697.69	long
44014.60, 50853.31) -	Loblaw	31	-5915.82%	-$2,627,629.39	short
44345.06	CNRail	36	442.06%	-$2,627,629.39	long
Talisman (112470.80,	Weston	30	7273.58%	$9,222,831.06	long
125903.92, 142069.49) -	Loblaw	31	-7788.72%	-$9,876,023.38	short
126587.03	CNRail	36	615.14%	$779,991.31	long
PetroCan (99314.10,	Weston	30	3773.63%	$4,570,524.78	long
119264.91, 145817.86) -	Loblaw	31	-3936.49%	-$4,767,783.94	short
120915.66	CNRail	36	262.87%	$318,376.72	long
Brookfield (932607.25,	Weston	30	6526.61%	$73,145,755.86	long
1106222.65, 1332031.70) -	Loblaw	31	-6963.85%	-$78,046,037.72	short
1119271.06	CNRail	36	537.24%	$6,021,013.05	long
Weston (51872.97,	CNRail	23	-2705.50%	-$1,474,060.81	short
54261.84, 56902.00) -	Loblaw	31	4583.63%	$2,497,337.10	long
54324.66	CNRail	36	-1778.13%	-$968,792.45	short
Shaw (48651.61,	Weston	30	8043.33%	$4,318,618.83	long
53376.73, 58952.61) -	Loblaw	31	-8638.92%	-$4,638,405.00	short
53589.42	CNRail	36	695.59%	$373,478.11	long
Loblow (139877.92,	CNRail	23	-768.24%	-$1,349,693.51	short
172111.18, 217560.79) -	Weston	30	1326.99%	$2,331,332.11	long
175415.27	CNRail	36	-458.75%	-$805,952.64	short

TABLE 11
RESULTS OF THE PASSIVE PENSION FUND IMMUNIZATION STRATEGY

Table 11 reports immunization results including characteristics of bonds (i.e., coupon rate, remaining time to maturity) included into the portfolio of pension fund assets, percentage investment weights, dollar investment amounts, and types of positions taken (i.e., long/short). We choose one longest-maturity bond from each maturity bucket, the short-term (0-5 years), medium-term (6 – 10 years), and long-term (>10 years), resulting in the portfolio investment strategy into the 10-year 5.53% coupon Canadian Tire bond, the 14-year 6.0% coupon Loblaw bond, and the 36-year 6.71% Canadian Rail bond.

Company (FPVL)-AFPVL	Bonds Selected	Maturity	Investment Weight	Investment Amount	Investment Strategy
Telus	CanTire	10.00	112.84%	$85,538.87	long
(68814.62,75355.32,83090	Loblaw	14.00	-14.00%	-$10,615.01	short
.81) - 75654.02	CNRail	36.00	1.16%	$879.87	long
CNRail (119723.64,	CanTire	10.00	83.28%	$109,868.25	long
131267.65,144750.75) -	Loblaw	14.00	16.53%	$21,813.93	long
131752.43	Loblaw	31.00	0.19%	$246.53	long
Encana (550096.56,	CanTire	10.00	38.08%	$234,868.24	long
613151.27, 686934.93) -	Loblaw	14.00	63.09%	$389,105.64	long
615833.50	CNRail	36.00	-1.16%	-$7,180.43	short
Enbridge (120837.31,	CanTire	10.00	144.68%	$185,928.93	long
128071.80, 136145.56) -	Loblaw	14.00	-47.33%	-$60,817.61	short
128281.62	CNRail	36.00	2.64%	$3,394.99	long
Can Tire (38497.71,	Talisman	9.00	41.01%	$18,215.90	long
44014.60, 50853.31) -	Loblaw	14.00	59.62%	$26,479.46	long
44345.06	CNRail	36.00	-0.63%	$26,479.46	short
Talisman (112470.80,	CanTire	10.00	88.14%	$111,757.65	long
125903.92, 142069.49) -	Loblaw	14.00	11.99%	$15,203.07	long
126587.03	CNRail	36.00	-0.13%	-$161.73	short
PetroCan (99314.10,	CanTire	10.00	11.64%	$14,098.95	long
119264.91, 145817.86) -	Loblaw	14.00	86.69%	$104,994.57	long
120915.66	CNRail	36.00	1.67%	$2,024.05	long
Brookfield (932607.25,	CanTire	10.00	70.92%	$794,871.76	long
1106222.65, 1332031.70) -	Loblaw	14.00	30.03%	$336,513.40	long
1119271.06	CNRail	36.00	-0.95%	-$10,653.96	short
Weston (51872.97,	CanTire	10.00	172.58%	$94,028.23	long
54261.84, 56902.00) -	Loblaw	14.00	-76.91%	-$41,901.34	short
54324.66	CNRail	36.00	4.33%	$2,356.96	long
Shaw (48651.61,	CanTire	10.00	105.93%	$56,878.13	long
53376.73, 58952.61) -	Loblaw	14.00	-6.73%	-$3,615.92	short
53589.42	CNRail	36.00	0.80%	$429.72	long
Loblow (139877.92,	CanTire	10.00	23.08%	$40,543.55	long
172111.18, 217560.79) -	Enbridge	13.00	74.25%	$130,438.44	long
175415.27	CNRail	36.00	2.68%	$4,703.97	long

APPENDIX 1
ARBITRAGE-FREE BOND VALUATION AND THE APPLICATION OF DURATION-CONVEXITY APPROACH IN CRISP AND FUZZY ANALYSIS

In crisp analysis, to find the time-specific intrinsic arbitrage-free value of the bond one uses an existing zero-coupon term structure of interest rates to discount all expected future bond cash flows. Accordingly, the value of the bond is given by the following formula:

$$V = \sum_{t=1}^{n} \frac{C_t}{(1+r_t)^t} = \left[\sum_{t=1}^{n} \frac{c \times F}{(1+r_t)^t}\right] + \frac{F}{(1+r_n)^n} = \left[\sum_{t=1}^{n-1} \frac{c \times F}{(1+r_t)^t}\right] + \frac{(c \times F)+F}{(1+r_n)^n}, \qquad \text{(A.1)}$$

where C_t is the cash flow produced by the bond, c is the bond's coupon rate, F is the face (par) value of the bond, and r_t is the t-period spot term structure rate. If the forward rate term structure is known, an alternative crisp formula may be used:

$$V = \sum_{t=1}^{n} \frac{C_t}{(1+r_1)} + \sum_{t=1}^{n} \frac{C_t}{\prod_{i=1}^{2}(1+r_i)} + \ldots + \sum_{t=1}^{n} \frac{C_t}{\prod_{i=1}^{n}(1+r_i)} = \left[\sum_{t=1}^{n-1} \frac{c \times F}{\prod_{i=1}^{n-1}(1+r_i)}\right] + \frac{(c \times F)+F}{\prod_{i=1}^{n}(1+r_i)}, \qquad \text{(A.2)}$$

With all parameters defined as before, except r_i that represents the i-period forward interest rate.

For the purposes of fuzzy analysis, we assume that each rate R_j is represented by a triangular fuzzy number. The fuzzy zero-coupon 1-year spot rate is assumed to be represented by R_j^0:

$R_j=(r_{jl}\ r_{jm}\ r_{jh})$ for j=1, 2, …, n and $R_j^0=(r_{jl}^0\ r_{jm}^0\ r_{jh}^0)$ for j=1, 2, …, n.

Then, we can write their α-cuts as follows:

$R_j(\alpha) =[r_{jl} +(r_{jm}- r_{jl})\, \alpha,\ r_{jh}-(r_{jh} - r_{jm})\, \alpha]$= for j=1, 2, …, n and $R_j^0=[r_{jl}^0 +(r_{jm}^0 - r_{jl}^0)\, \alpha,\ r_{jh}^0-(r_{jh}^0 - r_{jm}^0)\, \alpha]$ for j=1, 2, …, n.

Let $k_j(\alpha) = r_{jl} +(r_{jm}- r_{jl})\, \alpha$, and $s_j(\alpha) = r_{jh}-(r_{jh} - r_{jm})\, \alpha$ for $0 \le \alpha \le 1$, j=1,2,…, n and $k_j^0(\alpha) = r_{jl}^0 +(r_{jm}^0 - r_{jl}^0)\, \alpha$, and $s_j^0(\alpha) = r_{jh}^0-(r_{jh}^0 - r_{jm}^0)\, \alpha$ for $0 \le \alpha \le 1$, j=1,2,…, n.

Then, $\frac{1}{1+R_j(\alpha)} = \frac{1}{(1(+)(k_j(\alpha),s_j(\alpha))}$ and $\frac{1}{1+R_j^0(\alpha)} = \frac{1}{(1(+)(k_j^0(\alpha),s_j^0(\alpha))}$;

Suppose that C denotes our cash flows in each month found as a product of the coupon rate and the face (par) value of the bond. The last cash flow is found as a summation of the accrued coupon income for the last month and the face value of the bond. The coupon rate is usually not adjustable; therefore, it will enter the equation in the crisp form. Also, the fuzzy appropriate discount rate for period j is denoted by either $R_j(\alpha)=(k_j(\alpha),s_j(\alpha))$ or $R_j^0=(k_j^0(\alpha), s_j^0(\alpha))$. In order to apply the usual formula of multiplication, we have to have the condition for fuzzy numbers R_j and R_j^0 satisfied. Specifically, we need to have the left and right values of our α-cuts for different values of α belong to the positive domain: $R_j(\alpha)=(k_j(\alpha),s_j(\alpha)) \in \Re_+$ and $R_j^0= (k_j^0(\alpha), s_j^0(\alpha)) \in \Re_+$. In our example, these conditions are satisfied. Furthermore, in accordance with theory we can calculate the bond's price in 2 different ways.

In analogy with the formula in (A.1), when our spot rates (or yields to maturities are modelled as different fuzzy numbers for different periods to reflect the fact that the term structure is not flat) are fuzzy and different, our fuzzy counterpart would be:

If $R_j(\alpha) = (k_j(\alpha), s_j(\alpha)) = R_j(\alpha)$ for $0 \le \alpha \le 1 \ \forall\ j = 1,2,\ldots n$, then

$$FIVB(\alpha) = \left[\frac{C}{1+s_1(\alpha)}, \frac{C}{1+k_1(\alpha)}\right](+)\left[\frac{C}{[1+s_2(\alpha)]^2}, \frac{C}{[1+k_2(\alpha)]^2}\right](+)\ldots(+)\left[\frac{C}{[1+s_n(\alpha)]^n}, \frac{C}{[1+k_n(\alpha)]^n}\right] \quad \text{(A.3)}$$

In analogy with the formula in (2), when we have estimates of fuzzy spot rates and forward rates for each period, our fuzzy counterpart would be:

If $R_j(\alpha) = (k_j(\alpha), s_j(\alpha)) = R_j(\alpha)$ for $0 \le \alpha \le 1 \ \forall\ j = 1,2,\ldots n$, and we denote the product of two fuzzy then

$$FIVB(\alpha) = \left[\frac{C}{1+s_1(\alpha)}, \frac{C}{1+k_1(\alpha)}\right](+)\left[\frac{C}{\prod_{j=1}^{2}[1+s_j(\alpha)]}, \frac{C}{\prod_{j=1}^{2}[1+k_j(\alpha)]}\right](+)\ldots$$

$$(+)\left[\frac{C}{\prod_{j=1}^{n}[1+s_j(\alpha)]}, \frac{C}{\prod_{j=1}^{n}[1+k_j(\alpha)]}\right] \quad \text{(A.4)}$$

In this paper, we will focus on the first way to compute FIVB by employing formula (A.3) that can be re-written as follows:

$$FIVB(\alpha) = \left[\frac{C}{[1(+)(k_1(\alpha), s_1(\alpha))]}\right](+)\left[\frac{C}{[1(+)(k_2(\alpha), s_2(\alpha))](.)[1(+)(k_2(\alpha), s_2(\alpha))]}\right](+)\ldots$$

$$(+)\left[\frac{C}{[1(+)(k_n(\alpha), s_n(\alpha))]^n}\right] \quad \text{(A.5)}$$

Now, we can simplify it further by:

$$FIVB(\alpha) = \left[\frac{C}{1+s_1(\alpha)}, \frac{C}{1+k_1(\alpha)}\right](+)\left[\frac{C}{[1+s_2(\alpha)](.)[1+s_2(\alpha)]}, \frac{C}{[1+k_2(\alpha)](.)[1+k_2(\alpha)]}\right](+)\ldots$$

$$(+)\left[\frac{C}{[1+s_n(\alpha)]^n}, \frac{C}{[1+k_n(\alpha)]^n}\right] \quad \text{(A.6)}$$

Duration, as a weighted average time to maturity, is a measure of bond volatility to interest rate movements. In crisp analysis, duration is measured as follows:

$$D = \frac{\sum_{t=1}^{n} \frac{C_t}{(1+r_t)^t} \times t}{\sum_{t=1}^{n} \frac{C_t}{(1+r_t)^t}} = \frac{\sum_{t=1}^{n} PV(C_t) \times t}{B_0}, \qquad (A.7)$$

where B_0 is the sum of the present values of bond cash flows and r_t is the t-period spot interest rate.

Convexity measures the rate of change in duration as interest rates change. By adding convexity into the analysis, one is able to approximate the true bond price – yield relationship better. In crisp analysis, convexity is measured as follows:

$$C = \frac{\sum_{t=1}^{n} \frac{C_t}{(1+r_t)^t} \times (t^2 + t)}{\sum_{t=1}^{n} \frac{C_t}{(1+r_t)^t}} = \frac{\sum_{t=1}^{n} PV(C_t) \times (t^2 + t)}{B_0}, \qquad (A.8)$$

with all parameters defined as before.

The duration and convexity of the bond portfolio is defined as the weighted average durations/convexities of the bonds included in the portfolio. Formulas for portfolio duration and convexity are given below:

$$D_p = \sum_{i=1}^{N} D_i X_i \qquad (A.9)$$

$$C_p = \sum_{i=1}^{N} C_i X_i \qquad (A.10)$$

Where D_i is a duration of the bond i and X_i is a bond's i investment weight.

According to formula (A.7), fuzzy estimation of duration involves a separate estimation of fuzzy numerator and denominator values and dividing one fuzzy number by the other. The denominator represents the fuzzy intrinsic value of the bond itself denoted by $FIVB^d(\alpha)=(V^d(\alpha), F^d(\alpha))$ in equation (A.3), while the numerator can be represented as follows:

$$FIVB^n(\alpha) = (V^n(\alpha), F^n(\alpha)) = \left[\frac{C \times 1}{1+s_1(\alpha)}, \frac{C \times 1}{1+k_1(\alpha)}\right](+)\left[\frac{C \times 2}{[1+s_2(\alpha)]^2}, \frac{C \times 2}{[1+k_2(\alpha)]^2}\right](+)\ldots$$

$$(+)\left[\frac{C \times n}{[1+s_n(\alpha)]^n}, \frac{C \times n}{[1+k_n(\alpha)]^n}\right] \qquad (A.11)$$

The numerator for the fuzzy convexity can be represented similarly with the weights multiplied by (t^2+t) instead of t only. The division of one fuzzy number by the other is accomplished as follows:

$$FD(C)(\alpha) = (v(\alpha), f(\alpha)) = FIVB^n(\alpha)(:)FIVB^d(\alpha) = (V^n(\alpha), F^n(\alpha))(:)(V^d(\alpha), F^d(\alpha)) =$$

$$= \left[\frac{V^n(\alpha)}{F^d(\alpha)}, \frac{F^n(\alpha)}{V^d(\alpha)}\right] \qquad (A.12)$$

Since the appropriate fuzzy discount rate is denoted as $R_j(\alpha) = (k_j(\alpha), s_j(\alpha))$ and its approximate representation is $R_j = (r_{jl}, r_{jm}, r_{jh})$ for j = 1,2,…, n, our approximate fuzzy duration $AFD(\alpha)$ is given below :

$$AFD(\alpha) = (D_l, D_m, D_h) = \left(\left[\frac{\sum_{t=1}^{n} \frac{C}{(1+r_{jh})^t}(.)t}{\sum_{t=1}^{n} \frac{C}{(1+r_{jl})^t}} \right], \left[\frac{\sum_{t=1}^{n} \frac{C}{(1+r_{jm})^t}(.)t}{\sum_{t=1}^{n} \frac{C}{(1+r_{jm})^t}} \right], \left[\frac{\sum_{t=1}^{n} \frac{C}{(1+r_{jl})^t}(.)t}{\sum_{t=1}^{n} \frac{C}{(1+r_{jh})^t}} \right] \right) \quad \text{(A.13)}$$

Equivalently, the approximate fuzzy convexity, AFC(a), is:

$$AFC(\alpha) = (C_l, C_m, C_h) =$$

$$\left(\left[\frac{\sum_{t=1}^{n} \frac{C}{(1+r_{jh})^t}(.)(t^2+t)}{\sum_{t=1}^{n} \frac{C}{(1+r_{jl})^t}} \right], \left[\frac{\sum_{t=1}^{n} \frac{C}{(1+r_{jm})^t}(.)(t^2+t)}{\sum_{t=1}^{n} \frac{C}{(1+r_{jm})^t}} \right], \left[\frac{\sum_{t=1}^{n} \frac{C}{(1+r_{jl})^t}(.)(t^2+t)}{\sum_{t=1}^{n} \frac{C}{(1+r_{jh})^t}} \right] \right) \quad \text{(A.14)}$$

Pension benefit payments owed by the firm's sponsor to firm retirees are claims on future cash flows of the pension fund and the firm. The fuzzy pension liability stream (FPVL) can be represented by altering equation (A.3). If we represent our fuzzy cash outflows due at the end of period j as $O_j=(X_j(\alpha),Z_j(\alpha))$ for j=1,2,…,n, then

$$(X_j(\alpha)\ Z_j(\alpha))\,(.) \left[\frac{1}{1+s_j{}^0(\alpha)}, \frac{1}{1+k_j{}^0(\alpha)} \right] = \left[\frac{X_j(\alpha)}{1+s_j{}^0(\alpha)}, \frac{Z_j(\alpha)}{1+k_j{}^0(\alpha)} \right] \quad \text{(A.15)}$$

Now, we can transform the formula (A.15) to the simple fuzzy expression for finding FPVL as follows :

$$FPVL(\alpha) =$$

$$\left[\frac{X_1(\alpha)}{1+s_1{}^0(\alpha)}, \frac{Z_1(\alpha)}{1+k_1{}^0(\alpha)} \right] (+) \left[\frac{X_2(\alpha)}{\left[1+s_2{}^0(\alpha)\right]^2}, \frac{Z_2(\alpha)}{\left[1+k_2{}^0(\alpha)\right]^2} \right] (+)\ldots(+) \left[\frac{X_n(\alpha)}{\left[1+s_n{}^0(\alpha)\right]^n}, \frac{Z_n(\alpha)}{\left[1+k_n{}^0(\alpha)\right]^n} \right] \quad \text{(A.16)}$$