How the Declining Marginal Utility of Rewards Accentuates MR-MC Divergence at Profit Optimization

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In 2008 a mathematical proof refuting a long-standing principle in microeconomics was developed by the author. That economic principle says that firms, in order to optimize profit, should operate at a volume such that marginal revenue (MR) and marginal cost (MC) equate. The proof shows that, because volume is dependent on a key marginal cost (the rate of incentive pay), a firm's optimal volume will necessarily be less than where MR = MC. This paper extends the previous proof to assess the impact on a firm's optimal volume when the declining marginal utility associated with incentive pay is taken into account.

INTRODUCTION

An incentive pay function (a type of employee reward function) is a *coupling* function. Such a function exists when it serves to synthesize the interests of two or more parties in a cooperative engagement. Incentive pay is a cost to the firm and, simultaneously, a reward for the firm's employees, thus serving to integrate the interests of both parties.

This is a theoretical development paper[1] comparing the optimum incentive pay coupling function, when perceived reward (pay) utility is identical to the straight-line dollar incentive value, with the optimum incentive pay coupling function, when the perceived utility from the straight-line dollar function rises at a declining rate, "coming in below" the dollar function. In practice, this *declining* marginal utility function is the norm. The optimum incentive pay coupling function is defined as that function possessing a slope, or rate of change, that will allow the company to maximize profit.

OPTIMAL COUPLING FUNCTION WITH REWARD UTILITY EQUAL TO DOLLAR VALUE

In Figure 1 are graphs of a typical company revenue function (R) (Salvatore, 1996), typical incentive pay coupling function (r), and of a typical employee cost (sacrifice) function (c) as described by the Law of Escalating Marginal Sacrifice (Grant, 2004).

The linear reward (incentive pay) function in Figure 1 is the graph of the equation:

$$r = 30 + ax \qquad (1)$$

Where:

r is the firm's \$ labor cost (labor's reward),
x is the volume of output (units produced),
30 is \$30 per period--the labor payment at
zero production (a guaranteed minimal

payment the company has agreed to), and
a is the slope (rate of change) of the function, or the additional cost to the company (reward to the employees) for an additional unit produced.

This is primarily a pay-for-performance system (Helm, 2007) but \$30 is assured the employees regardless of performance. Incentive pay starts with the first unit produced. In Figure 1 we are assuming the utility (u) employees perceive to come from reward is the same as the dollar value of the reward (r), or r = u. *We seek the magnitude of the slope (a) of the reward function that will optimize profit*.

The employee, or labor, sacrifice (cost) function in Figure 1 is:

$$c = 20 + mx^2 \quad (2)$$

Where:

c is the amount (perceived negative value, or utility) of sacrifice,
m (and the 20) are constants helping describe the pattern of personal costs (sacrifices), and
x is as before.

This is a simple, sample function reflecting the Law of Escalating Marginal Sacrifice.

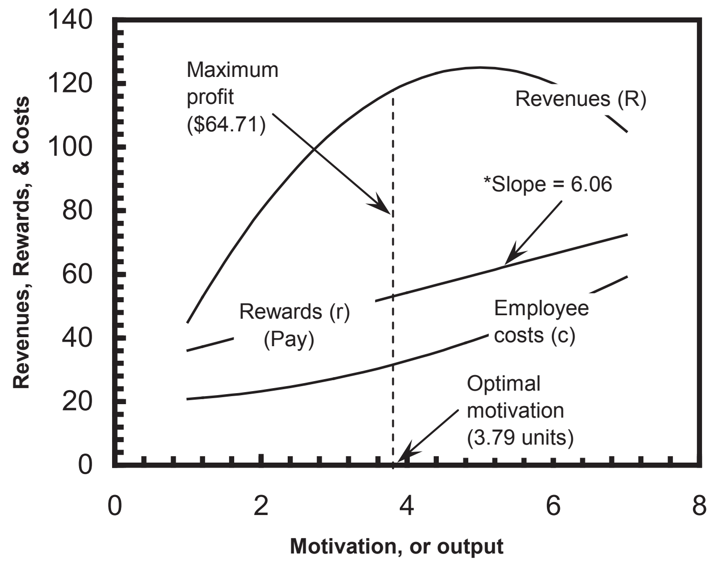
The labor force will seek to maximize its satisfaction, which it does when perceived marginal reward (Mr) and perceived marginal cost (or employee sacrifice) (Mc) equate--that is, when Mr = Mc, or $\partial r/\partial x = \partial c/\partial x$ (Chiang, 1984). From (1), $\partial r/\partial x = a$, and from (2), $\partial c/\partial x = 2mx$. Setting $\partial r/\partial x = \partial c/\partial x$ and solving for x, we have:

$$*x = a/2m \quad (3)$$

Where:

*x is the optimal volume, or that volume at which the employees perceive they will experience maximum satisfaction, and
a and m are as before.

FIGURE 1
OPTIMUM COUPLING FUNCTION



In this Figure the dollar value of the reward and the utility of the reward are the same. Here the optimal marginal reward, or coupling function slope, is 6.06. This generates an equality of employees' marginal cost (sacrifice) and marginal reward, or maximal satisfaction, at 3.79 units of motivation (or output), and a maximum possible profit of $64.71. No other reward function slope generates this high a profit. The assumption here is that perceived reward utility is congruent with the straight-line financial dollar incentive function (the reward curve in this figure). When the employee perceives less and less additional utility associated with added dollars, a declining marginal utility curve will exist below the straight-line dollar curve (r). Such a situation is addressed in Figure 2.

This is the level of output chosen by the workforce because it maximizes their satisfaction. Note this level is independent of the constant terms in the reward (r) and sacrifice (c) functions. It depends only on a, m, and the exponent of x.

Now, suppose the company faces the revenue function (R) in Figure 1:

$$R = 50x - 5x^2 \quad (4)$$

Where:

R is per period total revenue, and
x is as before.

Using (4) and (1), the company's profit function, excluding types of costs other than the above labor costs, is:

$$\text{Profit} = 50x - 5x^2 - 30 - ax \quad (5)$$

From (3), substituting for *x its equivalent, a/2m, and letting π represent profit, (5) converts to:

$$\pi = 50(a/2m) - 5(a/2m)^2 - 30 - a(a/2m) \quad (6)$$

This reduces to:

$$\pi = 25a/m - 1.25a^2/m^2 - 30 - a^2/2m \quad (7)$$

Profit is a clear function of the *slope*, a, of the company's cost function (employees' reward function) and of the variable, m, in the sacrifice function (employee cost function).

If the slope, a, is considered management's controllable variable, then from (7), to maximize profit, the company sets $\partial\pi/\partial a$ equal to zero and solves for the optimal slope, *a.

$$\partial\pi/\partial a = 25/m - 2.5a/m^2 - a/m = 0 \quad (8)$$

Solving for *a:

$$*a = 25m/(2.5 + m) \quad (9)$$

*a is the slope of the company's coupling function required to optimize profits. A greater or lesser slope will diminish profitability.

Now, if m = .8, as it does, for illustration purposes, in the employee cost function in Figure 1, and we use (9), we have:

$$*a = 25(.8)/2.5 + .8) = 6.06 \quad (10)$$

The slope (Mr, or $\partial r/\partial x$) of the Figure 1 coupling function is 6.06, which is the slope that optimizes profits for the organization. The *optimum reward (company cost), or coupling function*, referring back to (1), is thus:

$$r = 30 + 6.06x \quad (11)$$

From (3) and (10), the employee (labor) satisfaction maximizing volume is:

$$*x = *a/2m = 6.06/2(.8) = 3.79 \qquad (12)$$

This is the level of output (volume) the company can expect with the coupling function slope of 6.06. At this volume, the slope of the company revenue function (R) is from (4) and (12) above:

$$\partial R/\partial x = 50 - 10x = 50 - 10(3.79) = 12.1 \qquad (13)$$

The slopes of the company's cost (coupling) and revenue functions are *not* equal at the optimum volume $(6.06 \neq 12.1)$. *Marginal revenue (MR) does not equal marginal cost (MC, or Mr here) for profit maximization!*

How can this be? This defies contemporary core micro-economic thinking. The answer lies in the fact that we are restricting ourselves to a single linear reward (company cost) function, and, more importantly, *it is an attribute of costs, namely the slope of the company's cost function, that, when coupled with the employees' sacrifice curve, drives or determines volume, rather than the other way around*—the company's costs being determined by volume.

The level of employee motivation (output) required to generate the maximum profit possible here is, from (12), approximately 3.79 units. The profit is, repeating equation (5):

$$\pi = 50x - 5x^2 - 30 - ax \qquad (14)$$

Inserting *a and *x values from (10) and (12) we have:

$$*\pi = 50(3.79) - 5(3.79)^2 - 30 - 6.06(3.79) = \$67.41 \qquad (15)$$

This is the maximum possible profit, only obtained when the coupling function is, r = 30 + 6.06x, or exhibits a slope of 6.06.

OPTIMAL COUPLING FUNCTION WITH REWARD UTILITY RISING AT A DECLINING RATE

In Figure 2 we see graphs of the company revenue curve (same as in Figure 1), a coupling function (which is the company cost or employee dollar reward), a declining marginal employee utility curve, and the employee cost function (same as before). This figure illustrates the transformation of a straight line dollar reward function to its utility "equivalent". Figure 2 shows the coupling function possessing the optimal slope, the optimum motivation level, and the optimum company profit given the particular way in which utility is "spawned" from the dollar function—described in (19) below.

In Figure 2 company revenue is as before:

$$R = 50x - 5x^2 \qquad (16)$$

Employee cost is as before:

$$c = 20 + .8x^2 \qquad (17)$$

The *optimum coupling function,* however, is now:

$$r = 30 + 8.105x \qquad (18)$$

This equation for the optimum reward (company cost) function (r) takes the place of, r = 30 + 6.06x, which was the optimum equation when utility and dollar value were one and the same. This new coupling function occurs because the utility (u) function (a "derivative" of the coupling function) in Figure 2 reduces the rate of increase in real rewards by incorporating the factor, $- .4x^2$:

$$u = 30 + 8.105x - .4x^2 \quad (19)$$

Where:

u is the utility, or real perceived value,
of the dollar reward, and
x is as before.

DERIVATION OF OPTIMUM COUPLING AND UTILITY FUNCTIONS WHEN UTILITY RISES AT A DECLINING RATE

Let's take a look at how equations (18) and (19) were determined. The general linear dollar reward function (same as before) is:

$$r = 30 + ax \quad (20)$$

Suppose the general utility function is:

$$u = 30 + ax - .4x^2 \quad (21)$$

This is a sample utility function exhibiting declining marginal utility by adding the term, $- .4x^2$.

The given employee cost function (same as before) is:

$$c = 20 + .8x^2 \quad (22)$$

The given organizational revenue function is (as before):

$$R = 50x - 5x^2 \quad (23)$$

As always, employees are motivated to where $\partial u/\partial x = \partial c/\partial x$ (remember Mr = Mu when dollar value and utility were the same), or from (21) and (22), to where:

$$a - .8x = 1.6x \quad \text{or,} \quad {}^*x = a/2.4 \quad (24)$$

Now, repeating (14):

$$\pi = 50x - 5x^2 - 30 – ax \quad (25)$$

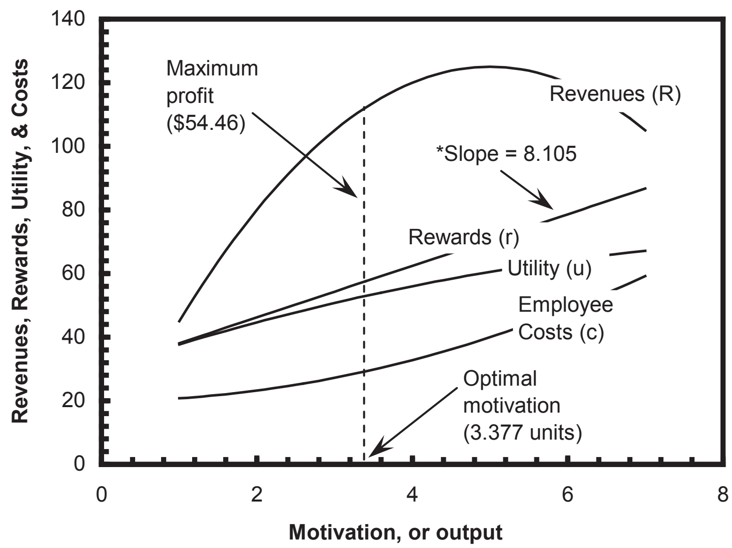
Or, inserting the optimal x (*x) from (24):

$$\pi = 50(a/2.4) - 5(a/2.4)^2 - 30 - a(a/2.4) \quad (26)$$

This reduces to:

$$\pi = 20.83a - 1.285a^2 – 30 \quad (27)$$

FIGURE 2
OPTIMUM COUPLING FUNCTION WITH DECLINING MARGINAL UTILITY



Illustrated here is the slope (magnitude, 8.105) of the employee reward function, or company cost function (a coupling function), which allows for the most profit possible. Such a slope creates the optimal motivation level of 3.377 units and a maximum profit of $54.46. The reward utility function (u) falls below the financial, straight-line incentive structure (r) causing the optimal motivation to be less than if the utility function coincided with the straight-line dollar curve. The lower motivation, due to the utility transformation (from the dollar curve, r, to the declining marginal utility curve, u) makes it necessary to raise the slope of the dollar curve to a higher value (to 8.105 from 6.06)—higher than when the utility and dollar curves coincide—to achieve the maximum possible profit. That maximum possible profit of $54.46 here is less than the maximum possible when the utility and dollar curve are identical as in Figure 1.

Taking the first partial and equating to zero for profit maximization yields:

$$\partial\pi/\partial a = 20.83 - 2.57a = 0 \quad (28)$$

Solving for the optimal a (*a):

$$*a = 8.105 \quad (29)$$

This is the value of the slope, a, you see in equations (18) and (19).

If the *a equals 8.105, the optimal x (*x) is, from (24):

$$*x = 8.105/2.4 = 3.377 \text{ units} \quad (30)$$

And, the optimal profit (*π) is now, from (25), (29) and (30):

$$*\pi = 50(3.377) - 5(3.377)^2 - 30 - 8.105(3.377) = \$54.46 \quad (31)$$

This is the maximum possible profit, only obtained when the coupling function is, r = 30 + 8.105x, or exhibits a slope of 8.105.

With recognition of the declining marginal utility of rewards, the organization must raise the slope of its coupling function above where it would have to be to maximize profits if there is no "separate" declining marginal utility function. The slope has to go from 6.06 to 8.105. But even though the slope of the coupling function is required to be greater, the level of motivation is less because it is the slope of the utility curve (not the dollar reward curve) and the slope of the employee cost (sacrifice) curve that determine motivation. Motivation under the straight-line (coupling) function is, from (12), 3.79 units; motivation under the declining marginal utility curve is, from (30), 3.377 units. Maximum possible profits are higher—$64.71, from (15), compared to $54.46, from (31)—without the declining marginal utility coming into play. In other words, the organization gets more "bang" out of its incentive dollars if there is no separate declining marginal utility function. Best if the perceived worth (utility) of dollars spent on financial reward stays constant and equates to the dollar value.

DIVERGENCES OF MARGINAL REVENUE FROM MARGINAL COST AT THE OPTIMAL VOLUMES

Classical microeconomics is built around the axiom that firms optimize profits when they produce up to that volume where marginal revenue (MR, or $\partial R/\partial x$) and marginal cost (here Mr, or $\partial r/\partial x$) equate. But this axiom is based on the assumption that management can choose directly a volume at which to operate.

If we assume that what management really does is *indirectly* affect output by designing the employee incentive (reward) structure, and that it is this reward structure that, in turn, influences employee choice about output, then we can see from the case illustrated in Figure 1 that $\partial R/\partial x$ and $\partial r/\partial x$ (marginal revenue and marginal cost to the company) do not equate at the volume that optimizes profit.

From (4) and (12):

$$MR = \partial R/\partial x = 50 - 10x = 50 - 10(3.79) = 12.1 \quad (32)$$

That is, the slope (MR) of the revenue function is 12.1 at the optimal volume.

Repeating (11) above:

$$r = 30 + 6.06x \quad (33)$$

So:

$$Mr = \partial r/\partial x = 6.06 \quad (34)$$

That is, the slope of the company's cost function (Mr, or $\partial r/\partial x$) is only 6.06 at the optimal volume. The difference between the firm's marginal revenue and the firm's marginal cost at the optimal volume (output) is 12.1 - 6.06 = 6.04, a significant difference given the functions involved.

But even more enlightening is what happens to the difference between MR and Mr, at the profit maximizing volume, when the declining slope of the utility function becomes a component. From (23) and (30):

$$\partial R/\partial x = 50 - 10x = 50 - 10(3.377) = 16.63 \quad (35)$$

That is, the slope of the revenue function increases to 16.63, from 12.1, at the new optimal volume.

Repeating (18) above:

$$r = 30 + 8.105x \quad (36)$$

So:

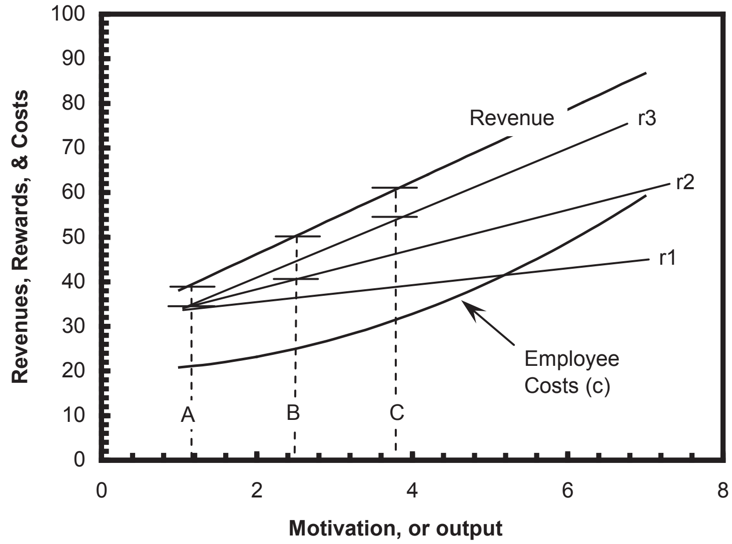
$$\partial r/\partial x = 8.105 \quad (37)$$

This says that the slope of the company's cost function is higher than when there is no declining marginal utility but still is only 8.105 at the optimal volume. The difference (16.63 - 8.105) is 8.525, an even greater divergence between the firm's marginal revenue and marginal cost at the profit optimizing volume.

The more realistic recognition of a declining marginal utility function exacerbates the difference between the firm's marginal revenue and marginal cost at the volume which optimizes profit. This suggests that in many situations, particularly when contingent financial rewards are a significant component of company cost, firms may want to operate substantially "away" from where marginal revenue and marginal cost are equal to one another—a major refutation of the classical position which states that firm's will maximize profit when operating at the volume where marginal revenue and marginal cost are equal.

Additional support for the fact that $MR \neq MC$, at the volume that optimizes profit, can be seen in Figure 3 where the firm experiences *linear* revenue and cost functions. At no volume can MR = MC, yet there exists an *optimal* volume.[2]

FIGURE 3
OPTIMUM COUPLING FUNCTION WITH LINEAR REVENUE & COSTS



Where dotted line A intersects the horizontal axis is approximately the motivation level at which the greatest difference exists between c and r1 (one of three reward curves illustrated). Where B intersects r2 is approximately the motivation level that maximizes the difference between c and r2. Where C intersects r3 is approximately the motivation level that optimizes the difference between c and r3. You can see that the difference between r2 and revenue on line B is greater than the difference between r1 and revenue on line A and greater than the difference between r3 and revenue on line C. Though r2 is not the *exact* profit maximizing reward (company cost) curve, it does yield higher profit than the other two reward curves suggesting that the true optimum reward curve lies somewhere between r1 and r3. Further increases in the slope of r (beyond r3) would increase motivation but result in lower profits. And certainly further decreases in the slope of r (below r1) would reduce motivation and decrease profits.

CONCLUSION

The overriding insight here is that if employees determine output as a response to rewards structured by management, rather than management choosing output levels directly, then the classical microeconomic axiom which states that firms will optimize profits at a volume where marginal revenue and marginal cost are equal is in error. And, in general, the greater the rate of declining marginal utility associated with a reward function, the greater will be the disparity between MR and MC (Mr here) at the volume of output that optimizes the firm's profit.[3]

ENDNOTES

1. This paper further advances insight into how individual employee behavior and the economic behavior of the firm are entwined. This theoretical integration is undergoing a long developmental process as new relationships between firm behavior and individual behavior are continuously being uncovered.

2. Additional evidence that firms should not seek to operate where MR = MC is to be found in the case of *linear* revenue and *linear* cost functions. Obviously the slopes of a linear revenue function and a linear cost function cannot be equal at any volume since the revenue function and cost function are, in all likelihood, not parallel. But there is an optimal volume, which optimizes profit, and a particular cost (reward) function, or "coupling function", slope that generates this optimal volume. This important phenomenon is absent from classical microeconomic and managerial economic theory. It occurs because employees ultimately determine volume based, at least in part, on their motivation as determined by incentives (defined by reward function slope) offered by management. Management does not *choose* volume *directly*. Management *influences* volume *indirectly* through the motivation of employees.

3. The reader can verify this increasing disparity by "plugging in" successively greater coefficients of the x squared term in the utility function, or by increasing the magnitude of the exponent of x in the last term of the utility function. Also, one may find it instructive to note that as utility functions exhibit more rapidly declining marginal utility, the optimal volumes will become less and less.

PERSPECTIVE

This work is part of a 30-plus year effort to merge microeconomic and employee motivation theory. The author has written dozens of articles and four books on the subject while teaching and researching in the related fields of managerial economics, operations management, organizational behavior (OB), and human resource management. More specifically, this manuscript presents the latest development in the author's ongoing attempts to relate OB and economic principles. All-in-all the major contribution of this paper lies in furtherance of the debunking of the long-held microeconomic axiom that says: firms should try to operate at a volume such that marginal revenue and marginal cost equate; doing such will maximize the firm's profit.

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